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SOME INEQUALITIES OF MEROMORPHIC p-VALENT FUNCTIONS ASSOCIATED WITH THE LIU-SRIVASTAVA OPERATOR

T.M. Seoudy[1], M.K. Aouf[1]

[1]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt
e-mail: vtms00@fayoum.edu.eg

Abstract. We derive several inequalities associated with a linear operator defined for a certain family of meromorphic p-valent functions. Also, we indicate relevant connections the various results present in this paper with those obtained in earlier work.

Keywords: analytic function, Schwarz function, generalized hypergeometric function, linear operator, Hadamard product, subordination.

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1. Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1,2,...\}), \tag{1}$$

which are analytic and meromorphic $p-$ valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U/\{0\}$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z)) (z \in U)$. In particular, if the function g is univalent in U, we have the equivalence (see [6]) :

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in \sum_p$ given by (1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z).$$

For real paremeters $a_1,...,a_q$ and $b_1,...,b_s\left(b_j \notin Z_0^- = \{0,-1,-2,...\}; j=1,...,s\right)$, we now define the generalized hypergeometric function ${}_qF_s\left(a_1,...,a_q;b_1,...,b_s;z\right)$ by (see [8])

$$ {}_qF_s\left(a_1,...,a_q;b_1,...,b_s;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_q)_k}{(b_1)_k...(b_s)_k}\frac{z^k}{k!}\left(q\le s+1;q,s\in N;z\in U\right), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\upsilon=0;\theta\in C^*=C\setminus\{0\}),\\ \theta(\theta+1)...(\theta+\upsilon-1) & (\upsilon\in N;\theta\in C).\end{cases} $$

Corresponding to the function $h_p\left(a_1,...,a_q;b_1,...,b_s;z\right)$ defined by

$$ h_p\left(a_1,...,a_q;b_1,...,b_s;z\right)=z^{-p}{}_qF_s\left(a_1,...,a_q;b_1,...,b_s;z\right), $$

we consider a linear operator $H_p\left(a_1,...,a_q;b_1,...,b_s\right):\sum_p\to\sum_p$, which is defined by the following Hadamard product (or convolution):

$$ H_p\left(a_1,...,a_q;b_1,...,b_s\right)f(z)=h_p\left(a_1,...,a_q;b_1,...,b_s;z\right)*f(z) \qquad (2) $$

or, equivalently, by

$$ H_p(a_1,...,a_q;b_1,...,b_s)f(z)=z^{-p}+\sum_{k=1}^{\infty}\frac{(a_1)_k...(a_q)_k}{(b_1)_k...(b_s)_k}\frac{a_{k-p}}{k!}z^{k-p}. \qquad (3) $$

If, for convenience, we write

$$ H_{p,q,s}\left(a_1\right)=H_p(a_1,...,a_q;b_1,...,b_s), \qquad (4) $$

then one can easily verify from the definition (2) or (3) that (see [4])

$$ z\left(H_{p,q,s}\left(a_1\right)f(z)\right)'=a_1H_{p,q,s}\left(a_1+1\right)f(z)-(a_1+p)H_{p,q,s}\left(a_1\right)f(z). \qquad (5) $$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [4] and Aouf [2]. In particular, for $s=1,q=2,a_1>0,b_1>0$ and $a_2=1$, we obtain the linear operator

$$ \ell_p(a_1,b_1)f(z)=H_p(a_1,1;b_1)f(z), \qquad (6) $$

which was introduced and studied by Liu and Srivastava [5].

We note that, for any integer $n>-p$ and $f\in\sum_p$,

$$ H_{p,2,1}(n+p,1;1)f(z)=D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z), \qquad (7) $$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [9], Yang [10], and Aouf [1].

To establish our main results we need the following lemma.

Lemma 1. [6] Let Ω be a set in the complex plane $\mathbb{C}$ and let c be a complex number satisfying $\Re(c) > 0$. Suppose that the function $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfies the condition:

$$\psi(ix, y; z) \notin \Omega \tag{8}$$

for all real $x, y \leq -\frac{|c-ix|^2}{2\Re(c)}$ and all $z \in U$. If the function $g(z)$ defined by $g(z) = c + c_1 z + c_2 z^2 + \ldots$ is analytic in U and if $\psi(g(z), zg'(z); z) \in \Omega$, then $\Re(g(z)) > 0$ in U.

In this paper, we shall derive some inequalities involving the linear operator $H_{p,q,s}(a_1)$ defined on meromorphic $p-$valent functions.

2. Inequalities involving the operator $H_{p,q,s}(a_1)$

Theorem 1. Let the function $f \in \sum_p$ defined by (1) satisfies the following inequality:

$$\Re\left\{\frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1)f(z)}\right\} < 1 + \frac{1-\alpha}{a_1} \quad (a_1 > 0; 0 \leq \alpha < 1; z \in U), \tag{9}$$

then

$$\Re\left\{\left(z^p H_{p,q,s}(a_1)f(z)\right)^{-\frac{1}{2\beta(1-\alpha)}}\right\} > 2^{-\frac{1}{\beta}} \quad (\beta \geq 1; z \in U).$$

The result is sharp.

Proof. Form (5) and (9), we have

$$\Re\left\{-\frac{z\left(H_{p,q,s}(a_1)f(z)\right)'}{H_{p,q,s}(a_1)f(z)}\right\} > p + \alpha - 1 (z \in U)$$

$$-\frac{1}{2(1-\alpha)}\Re\left\{\frac{z\left(H_{p,q,s}(a_1)f(z)\right)'}{H_{p,q,s}(a_1)f(z)} + p\right\} \prec \frac{z}{1-z}. \tag{10}$$

Let

$$g(z) = \left[z^p H_{p,q,s}(a_1)f(z)\right]^{-\frac{1}{2(1-\alpha)}}.$$

Then (10) may be written as

$$z[\ln g(z)]' \prec z\left[\ln\frac{1}{1-z}\right]'. \qquad (11)$$

Using a well-known result [7] to (11), we find that

$$g(z)=\left[z^p H_{p,q,s}(a_1)f(z)\right]^{-\frac{1}{2(1-\alpha)}} \prec \frac{1}{1-z},$$

that is, that

$$\left(z^p H_{p,q,s}(a_1)f(z)\right)^{-\frac{1}{2\beta(1-\alpha)}}=\left(\frac{1}{1-\omega(z)}\right)^{\frac{1}{\beta}}, \qquad (12)$$

where $\omega(z)$ analytic function in U with $w(0)=0$ and $|w(z)|<1 (z\in U)$.

According to $\Re\left(t^{\frac{1}{\beta}}\right)\geq\left(\Re(t)\right)^{\frac{1}{\beta}}$ $for \Re(t)>0$ and $\beta\geq 1$, (12) yields

$$\Re\left\{\left[z^p H_{p,q,s}(a_1)f(z)\right]^{-\frac{1}{2\beta(1-\alpha)}}\right\}=\Re\left[\left(\frac{1}{1-\omega(z)}\right)^{\frac{1}{\beta}}\right]\geq\left[\Re\left(\frac{1}{1-\omega(z)}\right)\right]^{\frac{1}{\beta}}>2^{-\frac{1}{\beta}}.$$

Further, we see that the result is sharp for the function

$$f(z)=z^{-p}+\sum_{k=1}^{\infty}\frac{(b_1)_k\ldots(b_s)_k(2\alpha-2)_k}{(a_1)_k\ldots(a_q)_k}z^{k-p}(z\in U^*).$$

This completes the proof of Theorem 1.

Remark 1. For $q=2, s=1$ and $a_2=1$, Theorem 1 yields the result which is obtained by Liu $[3, Theorem 1]$.

Putting $q=2, s=1, a_1=n+p$ $(n>-p, p\in\mathbb{N}), a_2=1$ and $b_1=1$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let the function $f\in\sum_p$ defined by (1) satisfy the following inequality:

$$\Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\}<1+\frac{1-\alpha}{n+p}\ (0\leq\alpha<1),$$

then

$$\Re\left\{\left(z^p D^{n+p-1}f(z)\right)^{-\frac{1}{2\beta(1-\alpha)}}\right\}>2^{-\frac{1}{\beta}}\ (\beta\geq 1).$$

The result is sharp for the function

$$f(z)=z^{-p}+\sum_{k=1}^{\infty}\frac{(2\alpha-2)_k}{(n+p)_k}z^{k-p}\ \left(z\in U^*\right).$$

Theorem 2. Let the function $f \in \sum_p$ defined by (1) satisfies the following inequality:

$$\Re\left\{(1-\lambda)\frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1)f(z)}+\lambda\frac{H_{p,q,s}(a_1+2)f(z)}{H_{p,q,s}(a_1+1)f(z)}\right\}<\alpha \tag{13}$$

$$(a_1>0;\alpha>1;0\le\lambda<a_1+1),$$

then

$$\Re\left\{\frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1)f(z)}\right\}<\beta,$$

where $\beta\in(\alpha,+\infty)$ is the positive root of the equation

$$2(a_1+1-\lambda)x^2+[3\lambda-2(a_1+1)\alpha]x-\lambda=0. \tag{14}$$

Proof. Let

$$\frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1)f(z)}=\beta+(1-\beta)g(z), \tag{15}$$

then $g(z)$ is analytic in U and $g(0)=1$. Differentiating (15) with respect to z and using (5) we deduce that

$$(1-\lambda)\frac{H_{p,q,s}(a_1+1)f(z)}{H_{p,q,s}(a_1)f(z)}+\lambda\frac{H_{p,q,s}(a_1+2)f(z)}{H_{p,q,s}(a_1+1)f(z)}=$$

$$=\beta+\frac{\lambda(1-\beta)}{a_1+1}+\frac{(1-\beta)(a_1+1-\lambda)}{a_1+1}g(z)+\frac{\lambda(1-\beta)}{a_1+1}\frac{zg'(z)}{\beta+(1-\beta)g(z)}=$$

$$=\psi\left(g(z),zg'(z)\right),$$

where

$$\psi(r,s)=\beta+\frac{\lambda(1-\beta)}{a_1+1}+\frac{(1-\beta)(a_1+1-\lambda)}{a_1+1}r+\frac{\lambda(1-\beta)}{a_1+1}\frac{s}{\beta+(1-\beta)r}. \tag{16}$$

Using (13) and (16), we have

$$\left\{\psi\left(g(z),zg'(z)\right):z\in U\right\}\subset\Omega=\{w\in\mathbb{C}:\Re(w)<\alpha\}.$$

Now, for all real $x,y\le-\frac{1+x^2}{2}$, we obtain

$$\Re\{\psi(ix,y)\}=\Re\left\{\beta+\frac{\lambda(1-\beta)}{a_1+1}+\frac{\lambda(1-\beta)}{a_1+1}\frac{\beta y}{\beta^2+(1-\beta)^2x^2}\right\}\ge$$

$$\geq \beta + \frac{\lambda(1-\beta)}{a_1+1} - \frac{\lambda(1-\beta)}{2(a_1+1)} \frac{1+x^2}{\beta^2+(1-\beta)^2 x^2} \geq$$
$$\geq \beta + \frac{\lambda(1-\beta)}{a_1+1} - \frac{\lambda(1-\beta)}{2\beta(a_1+1)} =$$
$$= \beta + \frac{\lambda(1-\beta)(2\beta-1)}{2\beta(a_1+1)} = \alpha,$$

where β is the positive root of (14).

Note that $0 \leq \lambda < a_1 + 1$ and $f(\alpha) = -\lambda(2\alpha-1)(\alpha-1) \leq 0$, then we have $\beta \in \alpha,+\infty)$. Hence for each $z \in U, \psi(ix, y) \notin \Omega$, by using Lemma 1, we get $\Re(g(z)) > 0$. This proves Theorem 2.

Remark 2. For $q = 2, s = 1$ and $a_2 = 1$, Theorem 2 yields the result which is obtained by Liu $[3, \mathit{Theorem}\, 2]$.

Putting $q = 2, s = 1, a_1 = n + p\ (n > -p, p \in \mathrm{N}), a_2 = 1$ and $b_1 = 1$ in Theorem 2, we obtain the following corollary.

Corollary 2. Let the function $f \in \sum_p$ defined by (1) satisfies the following inequality:

$$\Re\left\{(1-\lambda)\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} + \lambda \frac{D^{n+p+1} f(z)}{D^{n+p} f(z)}\right\} < \alpha \ (\alpha > 1; 0 \leq \lambda < n + p + 1).$$

Then

$$\Re\left\{\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}\right\} < \beta,$$

where $\beta \in (\alpha,+\infty)$ is the positive root of the equation

$$2(n+p+1-\lambda)x^2 + [3\lambda - 2(n+p+1)\alpha]x - \lambda = 0.$$

Theorem 3. Let $\lambda \geq 0, \alpha > 1$ and $a_1 > 0$. if the function $f, g \in \sum_p$ satisfies the following inequalities:

$$\Re\left\{\frac{H_{p,q,s}(a_1)g(z)}{H_{p,q,s}(a_1+1)g(z)}\right\} > \delta \ (0 \leq \delta < 1), \tag{17}$$

$$\Re\left\{(1-\lambda)\frac{H_{p,q,s}(a_1) f(z)}{H_{p,q,s}(a_1)g(z)} + \lambda \frac{H_{p,q,s}(a_1+1) f(z)}{H_{p,q,s}(a_1+1)g(z)}\right\} < \alpha, \tag{18}$$

then

$$\Re\left\{\frac{H_{p,q,s}(a_1) f(z)}{H_{p,q,s}(a_1)g(z)}\right\} < \frac{2\alpha a_1 + \lambda\delta}{2a_1 + \lambda\delta} \quad (z \in U).$$

Proof. Let $\beta = \frac{2\alpha a_1 + \lambda\delta}{2a_1 + \lambda\delta}$ and consider the function

$$\frac{H_{p,q,s}(a_1) f(z)}{H_{p,q,s}(a_1) g(z)} = \beta + (1-\beta)u(z), \tag{19}$$

where $u(z)$ is analytic in U and $u(0)=1$. Set $B(z) = \frac{H_{p,q,s}(a_1) g(z)}{H_{p,q,s}(a_1+1) f(z)}$, then $\Re(B(z)) > \delta$. Differentiating (19) with respect to z and using (5), we have

$$(1-\lambda)\frac{H_{p,q,s}(a_1) f(z)}{H_{p,q,s}(a_1) g(z)} + \lambda \frac{H_{p,q,s}(a_1+1) f(z)}{H_{p,q,s}(a_1+1) g(z)} =$$

$$= \beta + (1-\beta)u(z) + \frac{\lambda(1-\beta)}{a_1} B(z) z u'(z).$$

Let

$$\psi(r,s) = \beta + (1-\beta)r + \frac{\lambda(1-\beta)}{a_1} B(z) s,$$

then from (18), we deduce that

$$\left\{\psi\left(p(z), zp'(z)\right) : z \in U\right\} \subset \Omega = \{w \in \mathbb{C} : \Re(w) < \alpha\}.$$

Now, for all real $x, y \le -\frac{1+x^2}{2}$, we have

$$\Re\{\psi(ix,y)\} = \beta + \frac{\lambda(1-\beta)y}{a_1}\Re(B(z)) \ge \beta - \frac{\lambda(1-\beta)\delta}{2a_1}\left(1+x^2\right) \ge$$

$$\ge \beta - \frac{\lambda(1-\beta)\delta}{2a_1} = \alpha.$$

Hence for each $z \in U, \Re\{\psi(ix,y)\} \notin \Omega$. Thus by using Lemma 1, $\Re(u(z)) > 0$ in U. The Proof of Theorem 3 is completed.

Remark 3. For $q=2, s=1$ and $a_2 = 1$, Theorem 3 yields the result which is obtained by Liu $[3, \mathit{Theorem}\, 3]$.

Putting $q=2, s=1, a_1 = n+p\ (n > -p, p \in \mathbb{N}), a_2 = 1$ and $b_1 = 1$ in Theorem 3, we obtain the following corollary.

Corollary 3. Let $n > -p, \lambda \ge 0$ and $\alpha > 1$. İf the function $f, g \in \sum_p$ satisfies the following inequalities

$$\Re\left\{\frac{D^{n+p-1} g(z)}{D^{n+p} g(z)}\right\} > \delta\ (0 \le \delta < 1; z \in U),$$

$$\Re\left\{(1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}+\lambda\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right\}<\alpha\ (z\in U),$$

then

$$\Re\left\{\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right\}<\frac{2\alpha(n+p)+\lambda\delta}{2(n+p)+\lambda\delta}\ \ (z\in U).$$

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Liu-Srivastava operatoru ilə bağlı meromorf $p-$ valent funksiyalar üçün bəzi bərabərsizliklər

T.M. Seoudy, M.K. Aouf

XÜLASƏ

İşdə bir sinif meromorf $p-$valent funksiyalar üçün təyin olunmuş xətti operatorla bağlı bəzi bərabərsizliklər isbat olunmuşdur. Bundan başqa, alınmış nəticələrin əvvəlki işlərdə əldə edilmiş uyğun nəticələrlə əlaqələri göstərilmişdir.

Açar sözlər: analitik funksiya, Şvarts funksiyası, ümumiləşdirilmiş hyperhəndəsi funksiya, xətti operator, Adamar hasili, subardinasiya.

Некоторые неравенства для мероморфных p-валентных функций, связанных с оператором Лиу–Сривастава

Т.М. Сеоди, М.К. Аоф

РЕЗЮМЕ

Выведены неравенства, связанные с линейным оператором, определенным для некоторого класса мероморфных $p-$валентных функций. Кроме того, показывается связь результатов полученных в настоящей работе с результатами полученными в более ранних работах.

Ключевые слова: аналитическая функция, функция Шварца, обобщенная гипергеометрическая функция, линейный оператор, произведение Адамара, субординация.