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ON PERTURBATION OF X_d-BESSEL BASIS IN BANACH SPACES

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Abstract. In the paper, the perturbation of a basis in Banach spaces is studied. The notions of CB-space and X_d-Bessel basis in Banach spaces with respect to X_d space are introduced. Theorems on the perturbation of X_d-Bessel basis in Banach space with respect CB-space are established. In particular, known theorem on Riesz bases are generalized.

Keywords: CB-space, completeness, X_d-Bessel basis, X_d-Bessel system.

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1. Introduction

Sometimes, the basicity of the system in this or other space is established by means of the known perturbation theorems of the basis of this space. The Bari theorem on the perturbation orthonormalized basis in Hilbert space is known.

Theorem 1. ([1]) Let $\{\varphi_n\}_{n\in N}$ be an orthonormalized basis in Hilbert space H, the system $\{\psi_n\}_{n\in N} \subset H$ be ω-linearly independent in H, and

$$\sum_{n=1}^{\infty} \|\psi_n - \varphi_n\|_H^2 < +\infty .$$

Then, $\{\psi_n\}_{n\in N}$ forms isomorphic to $\{\varphi_n\}_{n\in N}$ basis in H, i.e. it is the Riesz basis.

Under some weaker condition this theorem was proved by Kato.

Theorem 2. ([2]) Let $\{\varphi_n\}_{n\in N}$ be an orthonormalized basis in Hilbert space H, the system $\{\psi_n\}_{n\in N} \subset H$. Then the system $\{\psi_n\}_{n\in N} \subset H$ is a basis in H if

$$\sum_{n=1}^{\infty} \left(\|\psi_n - \varphi_n\|_H^2 - \frac{|(\varphi_n - \psi_n, \psi_n)|^2}{\|\psi_n\|_H^2} \right) < 1 .$$

The theorems of Paley-Wiener, Krein-Rutman-Milman, Birkhoff-Rota, Kadetcs theorem-$1/4$ on perturbation in Banach spaces are known. Note that Kadets theorem-$1/4$ on Riesz basicity of the system $\{e^{i\lambda_n t}\}_{n\in Z}, T_m z$, in space $L_2(-\pi,\pi)$ is obtained from Paley-Wiener criterion under condition

$\sup_n |\lambda_n - n| < \frac{1}{4}$. The Levinson example ([3]) illustrates the exactness of the constant $1/4$. One may be acquainted with these or other facts of theory of bases, for instance in the monographs [4-8].

In Hilbert space H, any orthonormalized system $\{\varphi_n\}_{n\in N}$ is a Bessel system, i.e. there exists a constant $B > 0$ such that $\sum_{n=1}^{\infty} |(h, \varphi_n)|^2 \le B\ P$ for any $h \in H$. The Banach analogs of Bessel systems were studied in [9-12] and etc. The Bessel system in H is a special case of frame ([13-17]). The frames found numerous applications in many fields, such as signal processes, in processing of images, at data compression and etc. On stability of the frame, the papers [18-20] and others are known.

The present paper deals with perturbation of a basis in Banach spaces. The notion of CB-space, X_d-Bessel basis in Banach space with respect to CB-space X_d are introduced, the conditions of proximity, in definite sense, of the systems to X_d-Bessel basis under which the system forms an isomorphic basis, are found. In particular, the Bari known theorem on the Riesz is generalized.

2. Main results

Everywhere in the paper, X is a Banach space, $\tilde{X}$ is a Banach space of sequences $\{x_n\} \subset X$ with coordinate-wise linear operations and $\lim_{i\to\infty} \left\| \{x_n - \chi_{I_i}(n) x_n\} \right\|_{\tilde{X}} = 0$, such that, χ_{I_i} is a characteristic function of the set $I_i = \{n \in N : n \le i\}$ for any $i \in N$ (briefly KB-space), X_d is a KB-space of sequences of scalars.

X_d is said to be CB-space if

$$X_d^* = \left\{ \{c_n\} : (\{d_n\}, \{c_n\}) = \sum_{n=1}^{\infty} d_n \bar{c}_n, \{d_n\} \in X_d \right\}.$$

Let X_d be a CB-space. Say that X_d^* is normally subordinated to $\tilde{X}$ if for sequences $\{d_n\}$ and $\{x_n\}$ such that $|c_n| \le \|x_n\|_X$ from $\{x_n\} \in \tilde{X}$, it follows that $\{c_n\} \in X_d^*$ and $\|\{c_n\}\|_{X_d^*} \le \|\{x_n\}\|_{\tilde{X}}$.

In what follows, we need the following lemma.

Lemma 1. ([8]) Let X be a B-space, the system $\{\varphi_n\}_{n\in N} \subset X$ forms a basis in X, $F \in L(X)$ be Fredholm operator, the system $\{\psi_n\}_{n\in N} \subset X$ and $F(\varphi_n) = \psi_n$, $n \in N$. Then the following properties are equivalent:

a) $\{\psi_n\}_{n\in N}$ is complete in X;
b) $\{\psi_n\}_{n\in N}$ is minimal in X;
c) $\{\psi_n\}_{n\in N}$ is ω-linearly independent in X;
d) $\{\psi_n\}_{n\in N}$ forms a basis in X, isomorphic to $\{\varphi_n\}_{n\in N}$.

Let X_d be a KB-space, the system $\{\varphi_n\}\subset X$ and $\{\varphi_n^*\}_{n\in N}\subset X^*$ are biorthogonal. The pair $\left(\{\varphi_n^*\}_{n\in N},\{\varphi_n\}_{n\in N}\right)$ is said to X_d-Bessel in X with respect to X_d, if

1) $\{\varphi_n^*(x)\}_{n\in N}\in X_d$ for any $x\in X$;
2) there exists $B>0$ such that $\left\|\{\varphi_n^*(x)\}_{n\in N}\right\|_{X_d}\le B\,\|x\|_X$ for any $x\in X$.

The constant B is said to be a boundary of the X_d-Bessel pair $\left(\{\varphi_n^*\}_{n\in N},\{\varphi_n\}_{n\in N}\right)$. In the case of basicity of $\{\varphi_n\}_{n\in N}$ the X_d-Bessel pair $\left(\{\varphi_n^*\}_{n\in N},\{\varphi_n\}_{n\in N}\right)$ is called X_d-Bessel basis in X.

Theorem 4. Let X_d be a CB-space, $\tilde{X}$ be a KB-space and X_d^* be normally subordinated to $\tilde{X}$, the system $\{\varphi_n\}\subset X$ and $\left(\{\varphi_n^*\}_{n\in N},\{\varphi_n\}_{n\in N}\right)$ is X_d-Bessel basis in X with the boundary B, where $\{\varphi_n^*\}_{n\in N}$ is a biorthogonal system to $\{\varphi_n\}$, the system $\{\psi_n\}\subset X$ be such that $\{\varphi_n-\psi_n\}_{n\in N}\in\tilde{X}$. Then following properties are equivalent:

a) $\{\psi_n\}_{n\in N}$ is complete in X;
b) $\{\psi_n\}_{n\in N}$ is minimal in X;
c) $\{\psi_n\}_{n\in N}$ is ω-linearly independent in X;
d) $\{\psi_n\}_{n\in N}$ forms a basis in X is isomorphic to $\{\varphi_n\}$.

Proof. Show that $\sum_{n=1}^{\infty}\varphi_n^*(x)(\varphi_n-\psi_n)$ converges for any $x\in X$. For any $m,p\in N$ and $x\in X$, by the corollary of Hahn-Banach theorem exists $f_{m,p}\in X^*$: $\|f_{m,p}\|=1$ that

$$\left\|\sum_{n=m+1}^{m+p}\varphi_n^*(x)(\varphi_n-\psi_n)\right\|_X = f_{m,p}\left(\sum_{n=m+1}^{m+p}\varphi_n^*(x)(\varphi_n-\psi_n)\right).$$

We have

$$\left\|\sum_{n=m+1}^{m+p}\varphi_n^*(x)(\varphi_n-\psi_n)\right\|_X \le \left|\sum_{n=m+1}^{m+p}f_{m,p}(\varphi_n^*(x)(\varphi_n-\psi_n))\right| \le$$

$$\left|\sum_{n=m+1}^{m+p} f_{m,p}(\varphi_n-\psi_n)(\varphi_n^*(x))\right| = \left|\left(\left\{\chi_{I_{m+p}\backslash I_m}(n) f_{m,p}(\varphi_n-\psi_n)\right\}, \left\{\varphi_n^*(x)\right\}_{n\in N}\right)\right|$$

$$\le \left\|\left\{\chi_{I_{m+p}\backslash I_m}(n) f_{m,p}(\varphi_n-\psi_n)\right\}\right\|_{X_d^*} \times \left\|\left\{\varphi_n^*(x)\right\}_{n\in N}\right\|_{X_d} \le$$

$$\le$$

$$\left\|f_{m,p}\right\| \left\|\left\{\chi_{I_{m+p}\backslash I_m}(n)(\varphi_n-\psi_n)\right\}\right\|_{\tilde{X}} \left\|\left\{\varphi_n^*(x)\right\}_{n\in N}\right\|_{X_d} \le B\,\|x\|_X .$$

Consequently, $\sum_{n=1}^{\infty}\varphi_n^*(x)(\varphi_n-\psi_n)$ converges for any $x\in X$. So, the operator $T(x)=\sum_{n=1}^{\infty}\varphi_n^*(x)(\varphi_n-\psi_n)$, $x\in X$, is defined. Consider for each $m\in N$ the operator $T_m : X\to X$, by the formula $T_m(x)=\sum_{n=1}^{m}\varphi_n^*(x)(\varphi_n-\psi_n)$, $x\in X$. Obviously, that $T_m\in\sigma(X)$. Applying the inequality obtained about, we get

$$\left\|(T-T_m)(x)\right\|_X \le B\left\|\left\{\chi_{N\backslash I_m}(n)(\varphi_n-\psi_n)\right\}\right\|_{\tilde{X}} \|x\|_X .$$

Thus, $\left\|T-T_m\right\|\le B\left\|\left\{\chi_{N\backslash I_m}(n)(\varphi_n-\psi_n)\right\}\right\|_{\tilde{X}}$.

Therefore, $T=\lim_{m\to\infty}T_m\in\sigma(X)$. Then, the operator F given by the equality $F=I-T$ is a Fredholm operator, here I is an identity operator in X, and $F(x)=\sum_{n=1}^{\infty}\varphi_n^*(x)\psi_n$, $x\in X$. Obviously, $F(\varphi_n)=\psi_n$, $n\in N$. By Lemma 1, properties a) – d) are equivalent for the system $\{\psi_n\}_{n\in N}$. The theorem is proved.

Corollary 1. Let X_d be a CB-space, $\tilde{X}$ be a KB-space and X_d^* be normally subordinated to $\tilde{X}$, the system $\{\varphi_n\}_{n\in N}\subset X$ and $\left(\{\varphi_n^*\}_{n\in N},\{\varphi_n\}_{n\in N}\right)$ is X_d-Bessel basis in X, where $\{\varphi_n^*\}_{n\in N}$ is an biorthogonal system to $\{\varphi_n\}_{n\in N}$, $\{\psi_n\}_{n\in N}\subset X$ be ω-linearly independent and such that $\{\varphi_n-\psi_n\}_{n\in N}\in\tilde{X}$. Then $\{\psi_n\}_{n\in N}$ forms a basis in X is isomorphic to $\{\varphi_n\}$.

Remark. Note that if $X_d=l_2$ and $X=H$ is a Hilbert space and $\tilde{X}=l_2(H)=\left\{x=\{x_n\}\subset H : \|x\|_{l_2(H)}=\sum_{n=1}^{\infty}\|x_n\|^2<+\infty\right\}$, then orthonormaized basis in H is l_2-Bessel basis, and consequently, the Bari theorem is obtained from the Corollary 1.

Theorem 5. Let X_d be a CB-space, $\tilde{X}$ be a KB-space and X_d^* be normally subordinated to $\tilde{X}$, the system $\{\varphi_n\}_{n\in N} \subset X$ and $\left(\{\varphi_n^*\}_{n\in N}, \{\varphi_n\}_{n\in N}\right)$ is X_d-Bessel basis in X with the boundery B, where $\{\varphi_n^*\}_{n\in N}$ be biorthogonal system to $\{\varphi_n\}_{n\in N}$, $\{\psi_n\}_{n\in N} \subset X$ such that $\{\varphi_n - \psi_n\}_{n\in N} \in \tilde{X}$ and $\left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\tilde{X}} < \dfrac{1}{B}$. Then, $\{\psi_n\}_{n\in N}$ forms a basis in X which is isomorphic to $\{\varphi_n\}$.

Proof. From the proof of Theorem 4, we get that the series $\sum_{n=1}^{\infty} \varphi_n^*(x)(\varphi_n - \psi_n)$ converges for any $x \in X$. Define the operator T by the equality $T(x) = \sum_{n=1}^{\infty} \varphi_n^*(x)\psi_n$, $x \in X$. Obviously $T(\varphi_n) = \psi_n$, $n \in N$. Show the boundedness and bounded invariability of the operator T. For any $x \in X$ by the corollary of Hahn-Banach theorem, there exists $f_x \in X^*$ such that $\|f_x\| = 1$ and

$$f_x\left(\sum_{n=1}^{\infty} \varphi_n^*(x)(\varphi_n - \psi_n)\right) = \left\|\sum_{n=1}^{\infty} \varphi_n^*(x)(\varphi_n - \psi_n)\right\|_X .$$

Then,

$$\|(I-T)(x)\|_X = \left\|\sum_{n=1}^{\infty} \varphi_n^*(x)(\varphi_n - \psi_n)\right\|_X \le \left|\sum_{n=1}^{\infty} f_x(\varphi_n^*(x)(\varphi_n - \psi_n))\right| \le$$

$$\le \left|\left(\{f_x(\varphi_n - \psi_n)\}_{n\in N}, \{\varphi_n^*(x)\}_{n\in N}\right)\right| \le$$

$$\le \left\|\{f_x(\varphi_n - \psi_n)\}_{n\in N}\right\|_{X_d^*} \left\|\{\varphi_n^*(x)\}_{n\in N}\right\|_{X_d} \le$$

$$\le \|f_x\| \left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\tilde{X}} \left\|\{\varphi_n^*(x)\}_{n\in N}\right\|_{X_d} \le B \left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\tilde{X}} \|x\|_X .$$

Consequently, $\|I - T\| \le B \left\|\{\varphi_n - \psi_n\}_{n\in N}\right\|_{\tilde{X}}$, and the same token thereby $\|I - T\| < 1$. Therefore, the operator T is boundedly invertible. This completes the proof.

Example. Let X be a Banach space, $\tilde{X} = \left\{\{x_n\} \subset X : \|\{x_n\}\| = \sum_{n=1}^{\infty} \|x_n\| < +\infty\right\}$.

Suppose that $\{\varphi_n\}_{n\in N} \subset X$ is a basis in X such that $\sum_{n=1}^{\infty} |\varphi_n^*(x)| \le B\|x\|$ for any $x \in X$, where $\{\varphi_n^*\}_{n\in N} \subset X^*$ is a biorthogonal system to $\{\varphi_n\}_{n\in N}$, $B > 0$ is some constant. Let the system $\{\psi_n\}_{n\in N} \subset X$ such that $\{\varphi_n - \psi_n\}_{n\in N} \in \tilde{X}$. Then $\{\psi_n\}_{n\in N}$ forms a basis in X is isomorphic to $\{\varphi_n\}$ subject to one of the following conditions: a) $\{\psi_n\}_{n\in N} \subset X$ be ω-linearly independent in X; b) $\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\| < \frac{1}{B}$.

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Banax fəzalarında X_d - Bessel bazisinin həyəcanlanması

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XÜLASƏ

İşdə Banax fəzasında bazisin həyəcanlanması öyrənilir. X_d fəzasına nəzərən Banax fəzasında CB-fəzası və X_d Bessel bazisi anlayışı daxil edilir. Banax fəzasında CB-fəzasına nəzərən X_d Bessel bazisinin həyəcanlanması haqqında teoremlər isbat olunur. Xüsusi halda bazislər haqqında məlum Riss teoremi ümumiləşdirilir.

Açar sözlər: CB-fəza, tamlıq, X_d - Bessel bazisi, X_d -Bessel sistemi.

Возмущение X_d - базис Бесселя в банаховых пространствах

М.И. Исмаилов

РЕЗЮМЕ

В статье, изучается. возмущение базиса в банаховом пространстве. Вводится понятие CB-пространства и X_d-базис Бесселя в банаховом пространстве относительно пространства X_d. Доказаны теоремы о возмущении X_d-базис Бесселя в банаховом пространстве с относительно пространства CB. В частности, обобщается известная теорема о базисах Рисса.

Ключевые слова: CB-пространство, полнота, X_d-базис-Бесселя, X_d-система Бесселя.