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COMMON FIXED POINT THEOREM INVOLVING TWO MAPPINGS AND WEAKLY CONTRACTIVE CONDITION IN G-METRIC SPACES

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Abstract. The purpose of this paper is to establish a common fixed point theorem in the setting of G-metric spaces. Here, the notion of weak compatibility has been used to establish the existence of coincidence point and further altering distance function is employed to obtain the fixed point. We also provide an illustrative example in support of our results.

Keywords: Contraction, fixed point, coincidence point, altering distance function, weakly compatible mapping.

AMS Subject Classification: 54H25,47H10.

1. Introduction

Topological fixed point theory played an important role in the development of algebraic topology, during the early decades of the twentieth century. In 1922, Banach published his fixed point theorem, which is also known as Banach's Contraction Principle. This principle was founded on the underlying concept of Lipschitz mapping and thereafter this theorem has become a vigorous tool for studying nonlinear Volterra integral equations and nonlinear functional differential equations in Banach and other spaces.

The foundation concept of 2-metric space was established by Gahler [4, 5] and these spaces have subsequently been developed by Gahler and many others. Mustafa and Sims [10, 11] introduced a valid generalized metric space structure, which they call G-metric space. Some other papers dealing with G-metric spaces are [2, 3, 6, 12, 13, 14]. In 1984 Khan, Swaleh and Sessa [9] introduced a new type of contraction with the help of a control function, which they called altering distance function. After this a lot of results appeared in the literature which dealt with the altering distance function. The notion of weak compatibility was introduced by Jungck [8] which is a generalization of commuting mapping.

In [1] a common fixed point result has been proved using weak compatibility in metric space. The statement of the theorem is:

Theorem 1 [1]. Let f, g be two self maps of a metric space (X, d) satisfying

$$\psi\left(d\left(fx, fy\right)\right) \le \psi\left(d\left(gx, gy\right)\right) - \phi\left(d\left(gx, gy\right)\right) \tag{1}$$

for all x, y in X, where ψ and ϕ are continuous and non-decreasing mapping. If the range of g contains the range of f and g is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible then f and g have a unique common fixed point.

Further, in 2011 Hassen Aydi [3] proved a fixed point result for a self mapping on G-metric space satisfying (ψ,ϕ)-weakly contractive conditions.

Theorem 2 [3]. Let X be complete G-metric space. Suppose the map $T : X \to X$ satisfies for all $x, y, z \in X$,

$$\psi\left(G\left(Tx,Ty,Tz\right)\right) \le \psi\left(G\left(x,y,z\right)\right) - \phi\left(G\left(x,y,z\right)\right) \tag{2}$$

where ψ and ϕ are altering distance functions. Then T has a unique fixed point (say u) and T is G-continuous at u.

In this paper, we have used altering distance function to find a coincidence point result and also used weak compatibility to prove a common fixed point result.

2. Preliminaries

Definition 1. A mapping $\phi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied*:*

1. ϕ is continuous and non-decreasing
2. $\phi(t) = 0 \Leftrightarrow t = 0$.

Definition 2. Let X be a non-empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

1. $G(x,y,z) = 0$ if $x = y = z$
2. $G(x,x,y) > 0$ for all $x, y \in X$ with $x \ne y$
3. $G(x,x.y) \le G(x,y,z)$ for all $x, y, z \in X$ with $y \ne z$
4. $G(x,y,z) = G(x,z,y) = G(y,z,x) = \dots$
5. $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$.

Definition 3. Let (X,G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be limit of the sequence $\{x_n\}$ if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is G-convergent to x.

Definition 4. Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if for any $\varepsilon > 0$ there exist $k \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n,m,l \geq k$ that is $G(x_n, x_m, x_l) \to 0$ as $n,m,l \to \infty$.

Definition 5. A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 6 Let (X,G) be a G-metric space and f, g be two self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$.

Definition 7 Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Definition 8. A mapping $f:[0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

1. f is continuous and non-decreasing
2. $f(t) = 0 \Leftrightarrow t = 0.$

Lemma 1. Let f and g are weakly compatible self maps on a set X having a unique point of coincidence w (say), then w is the unique common fixed point of f and g.

3. Main Result

Theorem 3. Let f and g be two self maps on a complete G-metric space (X,G) satisfying,

$$\psi(G(fx, fy, fz)) \leq \psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)) \tag{3}$$

for all $x, y, z \in X$ and ϕ, ψ be altering distance function. If $f(X) \subseteq g(X)$ then f and g have a unique point of coincidence in X. Also if f and g are weakly compatible then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point, we can choose a point x_1 in X such that $f(x_0) = g(x_1)$ and continue in this way, we have

$$f(x_n) = g(x_{n+1}) \qquad n = 0,1,2,\ldots$$

From (3) we have,

$$\begin{aligned}\psi(G(gx_n, gx_{n+1}, gx_{n+1})) &= \psi(G(fx_{n-1}, fx_n, fx_n)) \\ &\leq \psi(G(gx_{n-1}, gx_n, gx_n)) - \phi(G(gx_{n-1}, gx_n, gx_n)) \\ &< \psi(G(gx_{n-1}, gx_n, gx_n)).\end{aligned} \tag{4}$$

Since ψ is non-decreasing. Therefore,

$$G\left(gx_n, gx_{n+1}, gx_{n+1}\right) < G\left(gx_{n-1}, gx_n, gx_n\right). \quad (5)$$

It follows that $\left\{G\left(gx_n, gx_{n+1}, gx_{n+1}\right)\right\}$ is a monotone decreasing sequence and hence converges to some $r \geq 0$.

Therefore, $\lim_{n\to\infty} G\left(gx_n, gx_{n+1}, gx_{n+1}\right) = \lim_{n\to\infty} r_n = r$.

Letting $n \to \infty$ in (4), we have, $\psi\left(r\right) \leq \psi\left(r\right) - \phi\left(r\right)$.

Using the continuity of ψ and ϕ, we have $\phi\left(r\right) = 0$. By using the property of ϕ we get $r = 0$ and thus we obtain,

$$\lim_{n\to\infty} G\left(gx_n, gx_{n+1}, gx_{n+1}\right) = 0. \quad (6)$$

Next we prove that, $\left\{gx_n\right\}$ is a G-Cauchy sequence. Let if possible, suppose that $\left\{gx_n\right\}$ is not a G-Cauchy sequence. Then there exist $\varepsilon > 0$ for which we can find two sub-sequences $\left\{gx_{m(i)}\right\}$ and $\left\{gx_{n(i)}\right\}$ of $\left\{gx_n\right\}$ with $n(i) > m(i) > i$ such that

$$G\left(gx_{n(i)}, gx_{m(i)}, gx_{m(i)}\right) \geq \varepsilon \quad (7)$$

and

$$G\left(gx_{n(i)-1}, gx_{m(i)}, gx_{m(i)}\right) < \varepsilon. \quad (8)$$

From (7) and (8), we have,

$$\begin{aligned}\varepsilon &\leq G\left(gx_{n(i)}, gx_{m(i)}, gx_{m(i)}\right)\\ &\leq G\left(gx_{n(i)}, gx_{n(i)-1}, gx_{n(i)-1}\right) + G\left(gx_{n(i)-1}, gx_{m(i)}, gx_{m(i)}\right)\\ &< \varepsilon + G\left(gx_{n(i)}, gx_{n(i)-1}, gx_{n(i)-1}\right).\end{aligned}$$

Also using the property of G-metric space, we obtain

$$\begin{aligned}0 \leq G\left(gx_{n(i)}, gx_{n(i)-1}, gx_{n(i)-1}\right) &= G\left(gx_{n(i)-1}, gx_{n(i)-1}, gx_{n(i)}\right)\\ &\leq G\left(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}\right).\end{aligned}$$

Letting $i \to \infty$ and using (6), we get, $G\left(gx_{n(i)}, gx_{n(i)-1}, gx_{n(i)-1}\right) \to 0$

Thus,

$$\lim_{i\to\infty} G\left(gx_{n(i)}, gx_{m(i)}, gx_{m(i)}\right) = \varepsilon. \quad (9)$$

Again, using the properties of G-metric space, we obtain,

$$G\left(gx_{n(i)},gx_{m(i)},gx_{m(i)}\right)\le G\left(gx_{n(i)},gx_{n(i)-1},gx_{n(i)-1}\right)$$
$$+G\left(gx_{n(i)-1},gx_{m(i)-1},gx_{m(i)-1}\right)\ +G\left(gx_{m(i)-1},gx_{m(i)},gx_{m(i)}\right)$$

and

$$G\left(gx_{n(i)-1},gx_{m(i)-1},gx_{m(i)-1}\right)\le G\left(gx_{n(i)-1},gx_{n(i)},gx_{n(i)}\right)$$
$$+G\left(gx_{n(i)},gx_{m(i)},gx_{m(i)}\right)\ +G\left(gx_{m(i)},gx_{m(i)-1},gx_{m(i)-1}\right).$$

Letting $i\to\infty$ in above inequalities, we obtain,

$$\lim_{i\to\infty}G\left(gx_{n(i)-1},gx_{m(i)-1},gx_{m(i)-1}\right)=\varepsilon. \tag{10}$$

Take $x=x_{n(i)-1}$ and $y=z=x_{m(i)-1}$ in (3) we have,

$$\psi\left(G\left(fx_{n(i)-1},fx_{m(i)-1},fx_{m(i)-1}\right)\right)=\psi\left(G\left(gx_{n(i)},gx_{m(i)},gx_{m(i)}\right)\right)$$
$$\le\psi\left(G\left(gx_{n(i)-1},gx_{m(i)-1},gx_{m(i)-1}\right)\right)-\phi\left(G\left(gx_{n(i)-1},gx_{m(i)-1},gx_{m(i)-1}\right)\right).$$

Taking $i\to\infty$, we get, $\psi(\varepsilon)\le\psi(\varepsilon)-\phi(\varepsilon)$ and this implies $\psi(\varepsilon)=0,$ that is $\varepsilon=0.$ This is a contradiction, therfore $\{gx_n\}$ is a G-Cauchy sequence and hence convergent. So there exists some $q\in X$ such that,

$$G\left(gx_n,gx_n,q\right)\to 0\ \text{and}\ G\left(gx_n,q,q\right)\to 0\ \text{as}\ n\to\infty \tag{11}$$

Now, we can find $p\in X$ such that $g(p)=q$. Again from (3), we have,

$$\psi\left(G\left(fp,fx_{n-1},fx_{n-1}\right)\right)=\psi\left(G\left(fp,gx_n,gx_n\right)\right)$$
$$\le\psi\left(G\left(gp,gx_{n-1},gx_{n-1}\right)\right)-\phi\left(G\left(gp,gx_{n-1},gx_{n-1}\right)\right).$$

Taking $n\to\infty$ and using (11) in above inequality, we get,

$$\lim_{n\to\infty}\psi\left(G\left(fp,gx_n,gx_n\right)\right)\le\psi(0)-\phi(0).$$

Hence we have,

$$\lim_{n\to\infty}\psi\left(G\left(fp,gx_n,gx_n\right)\right)=0 \tag{12}$$

Again, from properties of G-metric space

$$G\left(q,q,fp\right)=G\left(fp,q,q\right)\ \le G\left(fp,gx_n,gx_n\right)+G\left(gx_n,q,q\right).$$

By using (11) and (12), we have, $\lim_{n\to\infty}G\left(q,q,fp\right)=0\quad\Rightarrow\quad f(p)=q$

Hence $q=f(p)=g(p),$ that is q is the point of coincidence of f and g.

Assume that there is another point of coincidence t in X such that $t\ne q$ then there exist s in X, such that, $fs=gs=t$

Then again from (3), we obtain,

$$\psi\left(G\left(gp,gs,gs\right)\right)=\psi\left(G\left(fp,fs,fs\right)\right)$$
$$\leq\psi\left(G\left(gp,gs,gs\right)\right)-\phi\left(G\left(gp,gs,gs\right)\right)<\psi\left(G\left(gp,gs,gs\right)\right).$$

Which is a contradiction.

Thus f and g have a unique coincidence point in X. Now by using Lemma 1 we conclude that f and g have a unique fixed point in X

Corollary 1. Let X be a complete G-metric space. Suppose f and g be two self maps on G, satisfying

$$\psi\left(G\left(f^m x, f^m y, f^m z\right)\right)\leq\psi\left(G\left(gx,gy,gz\right)\right)-\phi\left(G\left(gx,gy,gz\right)\right)$$

for all $x,y,z\in X$ and ϕ,ψ be altering distance functions. Then f and g have a unique fixed point.

From Theorem 3.1 we conclude that f^m has a unique fixed point u (say). Since

$$fu = f\left(f^m u\right) = f^{m+1}u = f^m\left(fu\right)$$

Thus we have, fu is also a fixed point of f^m but by uniqueness we obtain $fu=u$

Example 1. *Let* $X=\left[0,1\right]\cup\left\{2,3,4,\ldots\right\}$ and Define

i. $G\left(x,y,z\right)=max\left\{\left|x-y\right|,\left|y-z\right|,\left|z-x\right|\right\}$, if $x,y,z\in[0,1]$ and at least $x\neq y$ or $y\neq z$ or $z\neq x$.

ii. $G\left(x,y,z\right)=x+y+z,$ if at least x or y or $z\notin[0,1]$ and at least $x\neq y, y\neq z$ or $z\neq x$.

iii. $G(x,y,z)=0$ if $x=y=z.$

Then $\left(X,G\right)$ is a complete G-metric space.

Define the mapping $\psi,\phi:[0,\infty)\to[0,\infty)$ as follows:

1. $\psi(t)=t$ if $0\leq t<1$ and $\psi(t)=t^2$ if $1\leq t$
2. $\phi(t)=\frac{1}{2}t^2$ if $0\leq t<1$ and $\phi(t)=\frac{1}{2}$ if $1\leq t$

Also, define the mapping $f,g:X\to X$ as follows:

1. $f(x)=x-\frac{1}{2}x^2$ if $0\leq x<1$ and $f(x)=x-1$ if $1\leq x$
2. $g(x)=x$ if $\leq x<1$ and $g(x)=x+1$ if $1\leq x$

Proof. Now we assume that $x>y>z$

Case I: Let $x\in[0,1]$ as $x>y>z$ then clearly $y,z\in[0,1]$.

By definition of G:

$$G\left(x,y,z\right)=x-z\in[0,1]$$

$$G(fx, fy, fz) = max\{|fx - fz|, |fy - fz|, |fz - fx|\}$$

$$= fx - fz = (x - z) - \frac{1}{2}(x^2 - z^2) \quad (13)$$

Again by definition of ϕ and ψ:

$$\psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)) = G(gx, gy, gz) - \frac{1}{2}G^2((gx, gy, gz)$$

$$= |gx - gz| - \frac{1}{2}|gx - gz|^2 = |x - z| - \frac{1}{2}|x - z|^2 \quad (14)$$

Therefore from (13) and (14) clearly we have (3).

Case II: $x \in \{3, 4, \ldots\}$ and as $x > y > z$

Then y and z may be in $\{3, 4, \ldots\}$ or in $[0,1]$.

Subcase I: Let $y \in \{3, 4, \ldots\}$ and $z \in \{3, 4, \ldots\}$.

Then $\psi(G(fx, fy, fz)) = \psi(x + y + z - 3) = (x + y + z - 3)^2$ (15)

and $\psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)) = (x + y + z + 3)^2 - \frac{1}{2}$ (16)

Then from (15) and (16), (3) is verified.

Subcase II: Let $y \in \{3, 4, \ldots\}$ and $z \in [0,1]$.

Then

$$\psi(G(fx, fy, fz)) = \psi(fx + fy + fz)$$

$$= \psi\left(x + y + z - \frac{z^2}{2} - 2\right) = \left(x + y + z - \frac{z^2}{2} - 2\right)^2 \quad (17)$$

and

$$\psi(G(gx, gy, gz)) - \phi(G(gx, gy, gz)) = \psi(gx + gy + gz) - \phi(gx + gy + gz)$$

$$= \psi(x + y + z + 2) - \phi(x + y + z + 2) = (x + y + z + 2)^2 - \frac{1}{2} \quad (18)$$

From (17) and (18) the inequality (3) is satisfied.

Case III: Let $x = 2$, since $x > y > z$ we have necessarily $y, z \in [0,1]$.

Therefore

$$\psi(G(fx, fy, fz)) = \psi(fx + fy + fz)$$

$$= \psi\left(x + y + z - \frac{y^2}{2} - \frac{z^2}{2} - 1\right) = \left(x + y + z - \frac{y^2}{2} - \frac{z^2}{2} - 1\right)^2 \quad (19)$$

and

$$\psi\left(G\left(gx,gy,gz\right)\right)-\phi\left(G\left(gx,gy,gz\right)\right)=\psi\left(gx+gy+gz\right)-\phi\left(gx+gy+gz\right)$$

$$=\left(x+y+z+1\right)^2-\frac{1}{2}\ . \qquad (20)$$

Hence from (19) and (20), we deduce that inequality (3) holds.
Thus the hypothesis of Theorem 3 are verified and we find that 0 is the unique fixed point of f and g in X .

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G –metrik fəzalarda iki inikas və zəif sıxılma şərtləri saxlayan ümumi tərpənməz nöqtə teoremi

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XÜLASƏ

Işin məqsədi G-metrik fəzalarda ümumi tərpənməz nöqtə teoremi isbat etməkdir. Burada zəif uyğunluq anlayışı üst-üstə düşmə nöqtəsinin varlığının isbatında, məsafə funksiyası isə tərpənməz nöqtənin tapılmasında istifadə olunur. Biz həm də alınmış nəticələr üçün bir illüstrativ misal veririk.

Açar sözlər: sıxılma, tərpənməz nöqtə, üst-üstə düşmə nöqtəsi, məsafə funksiyası, zəif uyğun inikaslar.

Общая теорема по неподвижной точке включающая двух отображений и условию слабо сжимаемости в G-метрических пространств

Вишал Гупта, Раман Диир

РЕЗЮМЕ

Целью данной работы является установление общей теоремы по неподвижной точке в G-метрических пространствах. Здесь понятие слабой совместимости было использовано для установления существования точки совпадения и далее функция расстояния используется для получения неподвижной точки. Мы также предоставляем иллюстративный пример для наших результатов.

Ключевые слова: сжатие, неподвижная точка, точка совпадения, функция расстояния, слабо совместимые отображения.