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GEOMETRY OF HYPERSURFACES OF A SEMI SYMMETRIC METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD

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Abstract. The purpose of the paper is to study the notion of CR-submanifold and the existence of some structures on a hypersurface of a semi symmetric semi metric connection in a quasi-sasakian manifold. We study the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of S.I. Goldberg-K. Yano [13]. We also discuss the integrability of distributions on M and geometry of their leaves.

Keywords: CR-submanifold, quasi-sasakian manifold, Semi-symmetric metric connection, Integrability conditions of the distributions.

AMS Subject Classification: 53D12, 53C05.

1. Introduction

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In [6, 15], Friedmann A. and Schouten J.A. introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)X - u(X)Y$$

where u is a 1-form. In [17], Yano K. studied some properties of semi-symmetric metric connections. Some properties of semi symmetric metric connections are studied in [1,8- 14].

The concept of CR-submanifold of a Kahlerian manifold has been defined by A. Bejancu [2]. Later, A. Bejancu and N. Papaghiue [3], introduced and studied the notion of semi-invariant submanifold of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kahlerian manifold. However the existence of the structure vector field implies some important changes.

The paper is organized as follows: In the first section, we recall some results and formulae for the later use. In the second section, we prove the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of S.I. Goldberg-K. Yano. The third section is concerned with integrability of distributions on M and geometry of their leaves.

2. Preliminaries

Let $\overline{M}$ be a real $(2n+1)$ dimensional differentiable manifold, endowed with an almost contact metric structure (f,ξ,η,g). Then we have from

(a) $f^2 = -I + \eta \otimes \xi,$

(b) $\eta(\xi) = 1$

I $\eta \circ f = 0;$ (1)

(d) $f(\xi) = 0;$

(e) $\eta(X) = g(X,\xi);$

(f) $g(fX, fY) = g(X,Y) - \eta(X)\eta(Y),$

for any vector field X,Y tangent to $\overline{M}$, where I is the identity on the tangent bundle $\Gamma\overline{M}$ of $\overline{M}$. Throughout the paper, all manifolds and maps are differentiable of class C^∞.We denote by $F(\overline{M})$ the algebra of the differentiable functions on $\overline{M}$ and by $F(E)$ the $F(\overline{M})$ module of the sections of a vector bundle E over $\overline{M}$.

The Niyembuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[fX,Y] + f[X,fY],$$
$$\forall X,Y \in \Gamma(T\overline{M})$$

and the fundamental 2-form is given by

$$\Phi(X,Y) = g(X,fY) \quad \forall X,Y \in \Gamma(T\overline{M}). \tag{2}$$

The curvature tensor field of $\overline{M}$, denoted by $\overline{R}$ with respect to the Levi-Civita connection $\overline{\nabla}$, is defined by

$$\overline{\mathrm{R}}(\mathrm{X},\mathrm{Y})\mathrm{Z} = \overline{\nabla}_{\mathrm{X}}\overline{\nabla}_{\mathrm{Y}}\mathrm{Z} - \overline{\nabla}_{\mathrm{Y}}\overline{\nabla}_{\mathrm{X}}\mathrm{Z} - \overline{\nabla}_{[\mathrm{X},\mathrm{Y}]}\mathrm{Z} \qquad \forall X,Y \in \Gamma(T\overline{M}) \tag{3}$$

Definition 1 (a) An almost contact metric manifold $\overline{M}$ (f,ξ,η,g) is called normal if

$$N_f(X,Y)+2d\eta(X,Y)\xi=0 \qquad \forall X,Y\in\Gamma(T\overline{M}). \tag{4}$$

Or equivalently (cf. [6])

$$(\overline{\nabla}_{fX} f)Y = f(\overline{\nabla}_X f)Y - g(\overline{\nabla}_X\xi,Y)\xi \quad \forall X,Y\in\Gamma(T\overline{M});$$

(b) The normal almost contact metric manifold $\overline{M}$ is called cosymplectic if $d\Phi = d\eta = 0$.

Let $\overline{M}$ be an almost contact metric manifold $\overline{M}$. According to [7] we say that $\overline{M}$ is a quasi-Sasakian manifold if and only if ξ is a killing vector field and

$$(\overline{\nabla}_X f)Y = g(\overline{\nabla}_{fX}\xi,Y)\xi-\eta(Y)\overline{\nabla}_{fX}\xi \qquad \forall X,Y\in\Gamma(T\overline{M}). \tag{5}$$

Next we define a tensor field F of type (1) by

$$FX=-\overline{\nabla}_X\xi \quad \forall X\in\Gamma(T\overline{M}). \tag{6}$$

Lemma 1. Let M be a quasi-Sasakian manifold. Then we have

(a) $(\overline{\nabla}_\xi f)X=0 \ \forall X\in\Gamma(T\overline{M})$;

(b) $foF=Fof$;

I $F\xi=0$;

(d) $g(FX,Y)+g(X,FY)=0 \qquad \forall X,Y\in\Gamma(T\overline{M})$; (7)

(e) $\eta oF=0$;

(f) $(\overline{\nabla}_X F)Y=\overline{R}(\xi,X)Y \qquad \forall X,Y\in\Gamma(T\overline{M})$;

The tersor field f defined on $\overline{M}$ an $f-$structure in sense of K. Yano that is $f^3+f=0$.

Definition 2. The quasi-Sasakian manifold $\overline{M}$ is said to be of rank $2p+1$ iff

$$\eta\wedge(d\eta)^p\neq 0 \text{ and } (d\eta)^{p+1}=0.$$

Example. Let (f,ξ,η,g) the almost contact metric structure defined by

$$\left[f_i^h\right]=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2y^1 & 2y^2 & 0 & 0 \end{bmatrix}$$

and

$$\left[f_i^{\,h}\right]=\begin{bmatrix} 1+4(y^1)^2 & 4y^1y^2 & 0 & 0 & 0 & 0 & -2y^1 \\ 4y^1y^2 & 1+4(y^2)^2 & 0 & 0 & 0 & 0 & -2y^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2y^1 & -2y^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi=(0,0,0,0,0,0,1)^t\ ,\quad \eta=dz-2y^1dx^1-2y^2dx^2.$$

It is easy to see that the above structure is a quasi Sasakian structure of rank 5.

Now we define a connection ∇ on M as

$$\overline{\nabla}_X Y=\nabla_X Y+\eta(Y)X-g(X,Y)\xi \tag{8}$$

such that $\overline{\nabla}_X g=0$ for any $X,\,Y\in TM$, where ∇_X is the Riemannian connection with respect to g on M. The connection $\overline{\nabla}$ is semi symmetric because

$$\mathrm{T(X,Y)}=\overline{\nabla}_X \mathrm{Y}-\overline{\nabla}_Y \mathrm{X}-[\mathrm{X,Y}]=\eta(\mathrm{Y})\mathrm{X}-\eta(\mathrm{X})\mathrm{Y}.$$

Using (8) in (5), we have

$$(\overline{\nabla}_X f)Y=g(\overline{\nabla}_{fX}\xi,Y)\xi-g(X,\,fY)\xi-\eta(Y)\overline{\nabla}_{fX}\xi-\eta(Y)\,fX \tag{9}$$

$$\overline{\nabla}_X\xi=-FX+X-\eta(X)\xi. \tag{10}$$

Let M be hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced levi-Civita connection on M and by $TM^{\perp}$ the normal vector bundle to M. The Gauss and Weingarton of hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold are

$$\text{(a)}\quad \overline{\nabla}_X Y=\nabla_X Y+B(X,Y)N; \tag{11}$$

$$\text{(b)}\quad \overline{\nabla}_X N=-AX,$$

where A is the shape operator with respect to the section N. It is known that

$$B(X,Y)=g(AX,Y)\quad \forall\, X,Y\in\Gamma(TM) \tag{12}$$

Because the position of the structure vector field with respect to M is very important we prove the following results.

Theorem 1. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. If the structure vector field ξ is normal to M then $\overline{M}$ is cosympletic manifold and M is totally geodesic immersed in $\overline{M}$.

Proof: Because $\overline{M}$ is quasi-Sasakian manifold, then it is normal and $d\phi=0$ ([4]). By direct calculation using (11) (b), we infer

$$d\eta(X,Y)=\frac{1}{2}\{(\overline{\nabla}_X\eta)(Y)-(\overline{\nabla}_Y\eta)(X)\}=\frac{1}{2}\{g(\overline{\nabla}_X\xi,Y)-g(\overline{\nabla}_Y\xi,X)\}$$

$$2d\eta(X,Y)=g(AY,X)-g(AX,Y)=0 \qquad \forall X,Y\in\Gamma(T\overline{M}); \qquad (13)$$

From (11) (b) and (13) we deduce

$$\begin{aligned}0&=d\eta(X,Y)=\frac{1}{2}\{(\overline{\nabla}_X\eta)(Y)-(\overline{\nabla}_Y\eta)(X)\}=\\&=\frac{1}{2}\{g(\overline{\nabla}_X\xi,Y)-g(\overline{\nabla}_Y\xi,X)\}=\\&=g(Y,\overline{\nabla}_X\xi)=-g(AX,Y)\quad \forall X,Y\in\Gamma(T\overline{M}),\end{aligned} \qquad (14)$$

which proves that M is totally geodesic. From (14) we obtain $\overline{\nabla}_X\xi=0$ $\forall X\in\Gamma(T\overline{M})$ By using (10), (7) (b) and (1) (d) from the above relation we state

$$-f(\overline{\nabla}_{fX}\xi)+X=\overline{\nabla}_X\xi, \qquad \forall X\in\Gamma(T\overline{M}), \qquad (15)$$

because $fX\in\Gamma(T\overline{M})$ $\forall X\in\Gamma(T\overline{M})$. Using (15) and the fact that ξ is not killing vector field, we deduce $d\eta\neq 0$.

Next we consider only the hypersurface which are tangent to ξ. Denote by $U=fN$ and from (1) (f), we deduce $g(U,U)=1$. Moreover it is easy to see that $U\in\Gamma(T\overline{M})$.Denote by $D^{\perp}=Span\{U\}$ the 1-dimensional distribution generated by U, and by D the orthogonal complement of $D^{\perp}\oplus\{\xi\}$ in TM. It is easy to see that

$$fD=D,\ fD^{\perp}\subseteq TM^{\perp},TM=D\oplus D^{\perp}\oplus\{\xi\}, \qquad (16)$$

where $\oplus$ denote the orthogonal direct sum. According with [2] from (16) we deduce that M is a CR-submanifold of $\overline{M}$.

Definition 3. A CR-submanifold M of a quasi-Sasakian manifold $\overline{M}$ is called CR-product if both distributions $D\oplus\{\xi\}$ and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifold of M.

Denote by P the projection morphism of TM to D and using the decomposion in (14) we deduce

$$X=PX+a(X)U+\eta(X)\xi \qquad \forall X\in\Gamma(TM), \qquad (17)$$

$$\begin{aligned}fX&=fPX+a(X)fU+\eta(fX)\xi=\\&=fPX-a(X)N.\end{aligned}$$

Since $U=fN,\quad fU=f^2N=-N+\eta(N)\xi=-N+g(N,\xi)\xi=-N$

where a is a 1-form on M defined by $a(X)=g(X,U)$ $\forall X\in\Gamma(TM)$. From (17) using (1) we infer

$$fX=tX-a(X)N, \qquad \forall X\in\Gamma(TM), \qquad (18)$$

where t is a tensor field defined by $tX = fPX \quad , \quad X \in \Gamma(TM)$
It is easy to see that

(a) $t\xi = 0;$ (19)

(b) $tU = 0.$

3. Induced structures on a hypersurface of a semi symmetric metric connection in a quasi-sasakian manifold

The purpose of this section is to study the existence of some induced structure on a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold. Let M be a hypersurface of a quasi-sasakian manifold $\overline{M}$. From (1) (a), (18) and (19) we obtain $t^3 + t = 0$, that is the tensor field t defines an f structure on M in sense of Yano K. [11]. Moreover, from (1) (a), (18), (19) we infer

$$t^2 X = -X + a(X)U + \eta(X)\xi \qquad \forall X \in \Gamma(TM). \tag{20}$$

Lemma 2. On a hypersurface of a semi symmetric non-metric connection M in a quasi-Sasakian manifold of a quasi-Sasakian manifold $\overline{M}$ the tensor field t satisfies

(a) $g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y),$ (21)

(b) $g(tX,Y) + g(X,tY) = 0 \;\; \forall X,Y \in \Gamma(TM)$.

Proof. From (1) (f), and (18) we deduce

$$g(X,Y) - \eta(X)\eta(Y) = g(fX, fY) = g(tX - a(X)N, tY - a(Y)N)$$

$$= g(tX,tY) - a(Y)g(tX,N) - a(X)g(N,tY) + a(X)a(Y)g(N,N)$$

$$= g(tX,tY) + a(X)a(Y),$$

$\forall X,Y \in \Gamma(TM)$

$$g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y)$$

(b) $$g(tX,Y) + g(X,tY) = g(fX + a(X)N, Y) + g(X, fY + a(Y)N)$$

$$= g(fX,Y) + a(X)g(N,Y) + g(X, fY) + a(Y)g(X,N)$$

$$= g(fX,Y) + g(X, fY) = 0.$$

And assertion (a) is proved. Assertion (b) follows from (20) and (21) (a).

Lemma 3. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. Then we have

(a) $FU = fA\xi$ (b) $FN = A\xi$ (c) $[U,\xi] = 0$ (22)

Proof. We take $X = U$ and $Y = \xi$ in (5)

$$f\left(\overline{\nabla}_U \xi\right) = -\overline{\nabla}_N \xi - N .$$

Then using (1) (a), (10), (11) (b), we deduce the assertion (a). The assertion (b) follows from (1) (a), (7) (b) and (11) (b) we derive

$$\overline{\nabla}_\xi U = (\overline{\nabla}_\xi f)N + f\overline{\nabla}_\xi N = -fA\xi = -FU = \overline{\nabla}_U \xi,$$
$$[U,\xi] = \overline{\nabla}_U \xi - \overline{\nabla}_\xi U = \overline{\nabla}_U \xi - \overline{\nabla}_U \xi = 0,$$

which prove assertion I. By using the decomposition $T\overline{M} = TM \oplus TM^{\perp}$, we deduce

$$FX = \alpha X - \eta(AX)N, \ \forall X \in \Gamma(T\overline{M}), \tag{23}$$

where α is a tensor field of type (1) on M, since $g(FX,N) = -g(X,FN) = -g(X,A\xi) = -\eta(AX)$, $X \in \Gamma(TM)$. By using (9), (10), (11), (18) and (20), we obtain

Theorem 2. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. Then the covariant derivative of a tensors t, a, η and α are given by

(a) $$(\nabla_X t)Y = g(FX, fY)\xi + \eta(Y)[\alpha tX - \eta(AX)U - 2fX] - a(Y)AX + B(X,Y)U \tag{24}$$

(b) $(\nabla_X a)Y = B(X,tY) + \eta(Y)\eta(AtX)$,

I $(\nabla_X \eta)Y = g(Y, \nabla_X \xi)$

(d) $(\nabla_X \alpha)Y = R(\xi, X)Y + B(X,Y)A\xi - \eta(AY)AX \quad \forall X,Y \in \Gamma(TM)$,

respectively, where R is the curvature tensor field of M.

From (9), (10), (19) (a) (b) and (24) (a) we get

Proposition 3.1. On a hypersurface of a semi symmetric metric connection M in a quasi-Sasakian manifold $\overline{M}$, we have

(a) $$\nabla_X U = -tAX - \eta(X)U + \eta(AtX)\xi, \tag{25}$$

(b) $B(X,U) = a(AX) \qquad \forall X \in \Gamma(TM)$.

Next we state

Theorem 3. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. The tensor field t is a parallel with respect to the Levi Civita connection ∇ on M iff

(a) $$AX = \eta(AX)\xi + a(AX)U \text{ and} \tag{26}$$

(b) $FX = \eta(AtX)U - \eta(AX)N + 2X - 2\eta(X)\xi, \qquad \forall X \in \Gamma(TM)$

Proof. Suppose that the tensor field t is parallel with respect to ∇, that is $\nabla t = 0$. By using (5) (a), we deduce

$$\eta(Y)[t\alpha(X) - \eta(AX)U - 2fX] - a(Y)AX + g(FX, fY)\xi + B(X,Y)U + 2g(fX,Y)\xi = 0 \qquad \forall X,Y \in \Gamma(TM). \tag{27}$$

Take $Y = U$ in (27) and using (11) (b), (12), (25) (b) we infer

$$\eta(U)[t\alpha(X) - \eta(AX)U - 2fX] - a(U)AX + g(FX, fU)\xi + 2g(fX,U)\xi + B(X,U)U = 0$$
$$\eta(U) = 0, \ a(U) = -1, \ g(X,N) = 0$$

$$-AX + g(FX, fU)\xi - 2g(X, fU)\xi + a(AX)U = 0$$
$$AX = g(FX, -N)\xi + a(AX)U$$
$$= g(X, FN)\xi + a(AX)U$$
$$= g(X, A\xi)\xi + a(AX)U = \eta(AX)\xi + a(AX)U.$$

And the assertion (25) (a) is proved. Next let $Y = fZ, Z \in \Gamma(D)$ in (27) and using (1) (f), (7) (b), (22) (a) (b), (26) (a), we deduce

$$g(X, FZ) = 0 \Rightarrow FX = \eta(AtX)U - \eta(AX)N + 2X - 2\eta(X)\xi \quad \forall X \in \Gamma(TM).$$

The proof is complete.

Proposition 2. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the assertions

(a) $(\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = -\eta(X)U$

(b) $(\nabla_X \eta)Y = 0 \Leftrightarrow \nabla_X \xi = 0 \quad \forall X, Y \in \Gamma(TM)$.

Proof. Let $X, Y \in \Gamma(TM)$ and using (12), (21) (b), (24) (b) and (25) (a) we obtain

$$g(\nabla_X U, Y) = g(-tAX + \eta(AtX)\xi - \eta(X)U, Y) =$$
$$= g(-tAX, Y) + \eta(AtX)g(\xi, Y) - \eta(X)g(U, Y) =$$
$$= g(AX, tY) + \eta(AtX)\eta(Y) - \eta(X)a(Y) =$$
$$= (\nabla_X a)Y - \eta(X)a(Y);$$
$$g(\nabla_X U + \eta(X)U, Y) = (\nabla_X a)Y$$
$$\Rightarrow \quad (\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = -\eta(X)U.$$

which proves assertion (a).

The assertion (b) is consequence of the fact that ξ is not a killing vector field.
According to Theorem 2 in [13], the tensor field

$$\bar{f} = t + \eta \otimes U - a \otimes \xi,$$

defines an almost complex structure on M. Moreover, from Proposition 2 we deduce

Theorem 4. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\bar{M}$. If the tensor fields t, a, η are parallel with respect to the connection ∇, then $\bar{f}$ defines a Kahler structure on M.

4. Integrability of distributions on a hypersurface of a semi symmetric metric connection in a quasi-sasakian manifold $\bar{M}$

In this section we established conditions for the Integrability of all distributions on a hypersurface of a semi symmetric metric connection M in a quasi-Sasakian manifold $\bar{M}$. From Lemma 3 we obtain

Corollary 1. On a hypersurface of a semi symmetric metric connection M in a quasi-Sasakian manifold $\overline{M}$ there exists a 2-dimensional foliation determined by the integral distribution $D^{\perp} \oplus \{\xi\}$

Theorem 5. Let M be a hypersurface of a semi symmetric non-metric connection in a quasi-Sasakian manifold $\overline{M}$. Then we have

(a) A leaf of $D^{\perp} \oplus \{\xi\}$ is totally geodesic submanifold of M if and only if

$$(1)\; AU = a(AU)U + \eta(AU)\xi \quad and$$
$$(2)\; FN = a(FN)U. \tag{28}$$

(b) A leaf of $D^{\perp} \oplus \{\xi\}$ is totally geodesic submanifold of $\overline{M}$ if and only if

$$(1)\; AU = 0 \quad and$$
$$(2)\; a(FX) = a(FN) = 0, \quad \forall X \in \Gamma(D).$$

Proof. (a) Let M^* be a leaf of integrable distribution $D^{\perp} \oplus \{\xi\}$ and h^* be the second fundamental form of the immersion $M^* \to M$. By using (1) (f), and (11) (b) we get

$$\begin{aligned} g(h^*(U,U),X) &= g(\overline{\nabla}_U U, X) = g(\overline{\nabla}_U (fN), X) = \\ &= g((\overline{\nabla}_U f)N, X) + g(f(\overline{\nabla}_U N), X) \\ &= -g(N, (\overline{\nabla}_U f)X) - g(\overline{\nabla}_U N, fX) \\ &= 0 - g(-AU, fX) = g(AU, fX) \quad \forall X \in \Gamma(D), \end{aligned} \tag{29}$$

and

$$\begin{aligned} g(h^*(U,\xi),X) &= g(\overline{\nabla}_U \xi, X) = g(-FU + U, X) = \\ &= g(FN, fX) + a(X) \quad \forall X \in \Gamma(D). \end{aligned} \tag{30}$$

Because $g(FN,N) = 0$ and $f\xi = 0$ the assertion (a) follows from (29) and (30).

(c) Let h_1 be the second fundamental form of the immersion $M^* \to M$. It is easy to see that

$$h_1(X,Y) = h^*(X,Y) + B(X,Y)N, \qquad \forall X,Y \in \Gamma(D^{\perp} \oplus \{\xi\}). \tag{31}$$

From (10) and (12) we deduce

$$\begin{aligned} g(h_1(U,U),N) &= g(h^*(U,U) + B(U,U)N, N) = \\ &= g(h^*(U,U),N) = g(\overline{\nabla}_U U, N), \\ &= -g(U, \overline{\nabla}_U N) = -g(U, -AU) = g(U, AU) = a(AU) \end{aligned} \tag{32}$$

$$\begin{aligned} g(h_1(U,\xi),N) &= g(h^*(U,\xi),N) = g(\overline{\nabla}_U \xi, N) = g(-FU + U, N) = \\ &= g(U, FN) = a(FN). \end{aligned} \tag{33}$$

The assertion (b) follows from (30)-(33).

Theorem 6. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. Then

(a) the distribution $D\oplus\{\xi\}$ is integrable iff

$$g(AfX+fAX,Y)=0,\ \ \forall X,Y\in\Gamma(D). \tag{34}$$

(b) the distribution D is integrable iff (34) holds and

$FX=\eta(AtX)U-\eta(AX)N$, (equivalent with $FD\perp D$) $\forall X\in\Gamma(D)$,

(c) The distribution $D\oplus D^{\perp}$ is integrable iff $FX=0,\forall X\in\Gamma(D)$.

Proof . Let $X,Y\in\Gamma(D)$. Since ∇ is a torsion free and ξ is killing vector field, we infer

$$g([X,\xi],U)=g(\overline{\nabla}_X\xi,U)-g(\overline{\nabla}_\xi X,U) \tag{35}$$
$$=g(\nabla_X\xi,U)+B(X,\xi)g(N,U)-g(\nabla_\xi X,U)-B(\xi,X)g(N,U)$$
$$=g(\nabla_X\xi,U)-g(\nabla_\xi X,U)=0,\qquad \forall X\in\Gamma(D),$$

Using (1) (a), (11) (a) we deduce

$$g([X,Y],U)=g(\overline{\nabla}_XY-\overline{\nabla}_YX,U)=g(\overline{\nabla}_XY-\overline{\nabla}_YX,fN)=$$
$$=g(\overline{\nabla}_YfX,N)-g(\overline{\nabla}_XfY,N) \tag{36}$$
$$=-g(f(\overline{\nabla}_XY),N)+g(f(\overline{\nabla}_YX),N)=-g(\overline{\nabla}_XfY,N)+g(\overline{\nabla}_YfX,N)$$
$$=-g(fY,\overline{\nabla}_XN)+g(fX,\overline{\nabla}_YN)=-g(AX,fY)-g(fX,AY)$$
$$=-g(fAX,Y)-g(AfX,Y)=-g(AfX+fAX,Y)\qquad\forall X,Y\in\Gamma(D).$$

Next by using (10) (7) (d) and the fact that ∇ is a metric connection we get

$$g([X,Y],\xi)=g(\overline{\nabla}_XY,\xi)-g(\overline{\nabla}_YX,\xi)=$$
$$=g(-\overline{\nabla}_X\xi,Y)-g(\overline{\nabla}_X\xi,Y)= \tag{37}$$
$$=2g(-\overline{\nabla}_X\xi,Y)=2g(FX-X+\eta(X)\xi,Y)=$$
$$=2g(FX,Y)-2g(X,Y)+2\eta(X)\eta(Y)\xi,\quad\forall X,Y\in\Gamma(D).$$

The assertion (a) follows from (35), (36) and assertion (b) follows from (35)-(37). Using (10) and (7) we obtain

$$g([X,U],\xi)=g(\overline{\nabla}_XU,\xi)-g(\overline{\nabla}_UX,\xi)=$$
$$=g(-\overline{\nabla}_X\xi,U)-g(\overline{\nabla}_X\xi,U) \tag{38}$$
$$=2g(FX-X,U)=2g(FX,U)-2g(X,U)+2\eta(X)g(\xi,U)\quad\forall X\in\Gamma(D)$$

Taking into account that

$$g(FX,N)=g(FfX,fN)=g(FfX,U),\qquad\forall X\in\Gamma(D). \tag{39}$$

The assertion I follows from (37) and (38).

Theorem 4. Let M be a hypersurface of a semi symmetric metric connection in a quasi-Sasakian manifold $\overline{M}$. Then we have

(a) the distribution D is integrable and its leaves are totally geodesic immersed in M if and only if

$FD \perp D$ and $AX = a(AX)U - \eta(AX)\xi, \quad \forall X \in \Gamma(D),$ (40)

(b) the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M if and only if

$AX = a(AX)U$, $X \in \Gamma(D)$ and $FU = 0$ (41)

the distribution $D \oplus D^{\perp}$ is integrable and its leaves are totally geodesic immersed in M if and only if $FX = 0$ $X \in \Gamma(D)$.

Proof. Let M_1^* be a leaf of integrable distribution D and h_1^* the second fundamental form of immersion $M_1^* \to M$. Then by direct calculation we infer

$$g(h_1^*(X,Y),U) = g(\overline{\nabla}_X Y,U) = -g(Y,\nabla_X U) = -g(AX,tY), \tag{42}$$

and

$$g(h_1^*(X,Y),\xi) = g(\overline{\nabla}_X Y,\xi) = g(FX,Y) - g(X,Y) + \eta(X)\eta(Y), \quad \forall X,Y \in \Gamma(D). \tag{43}$$

Now suppose M_1^* is a totally submanifold of M . Then (4.13) follows from (42) and (43). Conversely suppose that (40) is true. Then using the assertion (b) in Theorem 7 it is easy to see that the distribution D is integrable. Next the proof follows by using (42) and (43). Next, suppose that the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic submanifolds of M . Let M_1 be a leaf of $D \oplus \{\xi\}$ and h_1 the second fundamental form of immersion $M_1 \to M$. By direct calculations, using (10), (11) (b), (21) (b) and (25) (c), we deduce

$$g(h_1(X,Y),U) = g(\overline{\nabla}_X Y,U) = -g(AX,tY), \quad \forall X,Y \in \Gamma(D), \tag{44}$$

and

$$g(h_1(X,\xi),U) = g(\overline{\nabla}_X \xi,U) = g(-FU + U - \eta(U)\xi, X) = g(FU,X) - g(U,X), \quad \forall X \in \Gamma(D) \tag{45}$$

Then the assertion (b) follows from (4.12), (4.17), (4.18) and the assertion (a) of Theorem 7. Next let $\overline{M}_1$ a leaf of the integrable distribution $D \oplus D^{\perp}$ and $\overline{h}_1$ the second fundamental form oh the immersion $\overline{M}_1 \to M$. By direct calculation we get

$$g(\overline{h}_1(X,Y),\xi) = g(FX,Y) - g(X,Y) + \eta(X)\eta(Y), \tag{46}$$

$\forall X \in \Gamma(D), Y \in \Gamma(D \oplus D^{\perp})$.

The assertion I follows from (7) I, (39) and (46).

References

1. Ahmad M, Haseeb A, Jun J. B, Rahman S, On Almost r-paracontact Riemannian Manifold with a Certain Connection, Commun. Korean Math. Soc., Vol.25, No.2, 2010, pp.235-243.
2. Bejancu A., CR-submanifold of a Kahler manifold, I Proc. Amer. Math. Soc., Vol.69,1978, pp.135-142.
3. Bejancu A., Papaghiuc N. Semi-invariant submanifolds of a Sasakian manifold. An St. Univ. Al I Cuza Iasi supl., Vol.XVII, 1, I-a, 1981, pp.163-170.
4. Blair D.E. The theory of quasi Sasakian structures, J. Diff. Geometry, 1967, I: 331-345.
5. Calin C., Contributions to geometry of CR-submanifold, Thesis, University Al. I. Cuzalasi, Romania, 1998.
6. Friedmann A., Schouten J.A., Uber die Geometric der halbsymmetrischen Ubertragung, Math. Z, Vol.21,1924, pp.211–223.
7. Goldberg S.I., Yano K., On normal globally framed f-manifold, Tohoku Math. J., Vol.22, 1970, pp.362-370.
8. Jae-Bok Jun, Mobin Ahmad. Hypersurfaces of almost r-paracontact Riemannian manifold endowed with semi-symmetric metric connection, Bull. Korean Math. Soc., Vol.46, 2009, pp.895-903.
9. Rahman S, Ahmad A., Transversal Hypersurfaces of Almost Hyperbolic contact manifolds endowed with canonical semi symmetric semi metric connection, Journal of Qafqaz University, Vol.34, No.2, 2012, pp.25-33.
10. Rahman S, Shafiullah, Geometry of Hypersurfaces of a Semi Symmetric Semi Metric Connection in a Quasi-Sasakian Manifold, Journal of Purvanchal Academy of Sciences, Vol.17, 2011, pp.231-242.
11. Rahman S, Some Properties of Hyperbolic contact Manifold in a Quasi Sasakian Manifold, Turkic World Mathematical Society Journal of Applied and Engineering Mathematics, Vol.1, No.1, 2011, pp.41-48.
12. Rahman S. Transversal Hypersurfaces of a Lorentzian Para-Sasakian Manifold with a Semi-Symmetric Metric Connection, Journal of Qafqaz University Mathematics and Computer Science, Vol.1, No.1, 2013, pp.137-144.
13. Rahman S. Transversal Hypersurfaces of Almost Hyperbolic Contact Manifolds Endowed with Semi Symmetric Metric Connection, World Applied Programming, Vol.2, No.3, 2012, pp.195-201.
14. Rahman S., Transversal Hypersurfaces of Almost Hyperbolic contact manifolds endowed with semi symmetric non metric connection, Turkic World Mathematical Society Journal of Pure and Applied Mathematics, Vol.3, No.2, 2012, pp.210-219.
15. Schouten JA. Ricci calculus. Springer, 1954.

16. Yano K. On semi-symmetric metric connection, Rev. Roumaine Math. Pures. Appl., Vol.15, 1970, pp.1579–1586.
17. Yano K. On structure defined by a tensor field f of type (1, 1), satisfying f3 + f = 0. Tensor NS, Vol.14,1963, pp.99-109.

Kvazi-sasakian çoxobrazlısında yarımsimmetrik metrik əlaqələrin hipersəthlərinin həndəsəsi

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XÜLASƏ

Məqalədə CR – altçoxobrazlı anlayışı tətbiq edilir və simmetrik yarımmetrik əlaqələrin hipersəthində müəyyən strukturların varlığı məsələsi araşdırılır.

Açar sözlər: CR – altçoxobrazlı, kvazi-sasakian çoxobrazlı, yarı simmetrik metrik əlaqə, paylanmanın inteqrallanması şərtləri.

Геометрия гиперповерхностей полусимметрических связей на квази-сасакиан многообразии

Шамсур Рахман

РЕЗЮМЕ

В работе исследуется понятие CR – подмнообразии и существование некоторых структур по гиперповерхности симметричных полуметрик связей.

Ключевые слова: CR- полумногообразие, квази-сасакиан многообразие, полусимметрическая связь, условие интегрирования распределения.