GEODESİCS OF THE SYNECTİC METRİC

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Abstract. The main purpose of the paper is to investigate geodesics on the tangent bundles $T(M_n)$ of the Riemannian manifold with respect toy the Levi-Civita connection of the synectic metric ${}^S g = {}^C g + {}^V a$, where ${}^C g$ -complete lift of the Riemannian metric, ${}^V a$ - vertical lift of the symmetric tensor field *a*.

Keywords: geodesics, synectic metric, vertical and complete lift.

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1. Introduction

Let M_n be an *n*-dimensional differentiable manifold of class C^∞ and $T_p(M_n)$ the tangent space at a point *P* of M_n, that is, the set of all tangent vectors of M_n at *P*. Then the set

$$T(M_n) = \bigcup_{P \in M_n} T_p(M_n),$$

is by definition, tangent bundle over the manifold M_n [2]. We denote by $\Im_q^p(M_n)$ the set of all tensor fields of type (p,q) in M_n and by $\pi : T(M_n) \to M_n$ the natural projection over M_n. For $U \subset M_n$, $\left(x^i, x^{i'}\right)$, $i = 1,...n$ and $i' = n+1,...,2n$ are local coordinates in a neighborhood $\pi^{-1}(U) \subset T(M_n)$. If $\left\{U', x^{i'}\right\}$ is another coordinate neighborhood in M_n containing the point $P = \pi\left(\widetilde{P}\right)$ ($P \in U$ and $\widetilde{P} \in T_p(M_n)$), then $\pi^{-1}\left(U'\right)$ contains $\widetilde{P}$ and the induced coordinates of $\widetilde{P}$ with respect to $\pi^{-1}\left(U'\right)$ will be given by

$$x^{i'} = x^{i'}\left(x^i\right)$$

$$x^{\bar{i}'} = v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i = x^{\bar{i}} \frac{\partial x^{i'}}{\partial x^i}$$

and the Jacobian is given by the matrix

$$A=\left(\frac{\partial x^{I'}}{\partial x^{I}}\right)=\begin{pmatrix}\frac{\partial x^{i'}}{\partial x^{i}} & \frac{\partial x^{i'}}{\partial x^{\bar{i}}}\\ \frac{\partial x^{\bar{i}'}}{\partial x^{i}} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{i}}}\end{pmatrix}=\begin{pmatrix}\frac{\partial x^{i'}}{\partial x^{i}} & 0\\ \frac{x^{\bar{s}}\partial^2 x^{i'}}{\partial x^{s}\partial x^{i}} & \frac{\partial x^{i'}}{\partial x^{i}}\end{pmatrix}.$$

Let M_n be a Riemannian manifold with metric g whose components in a coordinate neighborhood U are g_{ji} and denote by Γ^h_{ji} the Christoffel symbols formed with g_{ji}. In the neighborhood $\pi^{-1}(U)$ of $T(M_n)$, U being a neighborhood of M_n, we put

$$\delta y^h = dy^h + \Gamma^h_i dx^i$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)\subset T(M_n)$, where

$$\Gamma^h_i = y^j\,\Gamma^h_{ji}.$$

Suppose that there is given the following Riemannian metric

$${}^s\tilde{g}_{CB}dx^C dx^B = a_{ji}dx^j dx^i + 2g_{ji}dx^j\delta y^i \qquad (1)$$

in the tangent bundle in $T(M_n)$ over a Riemannian manifold M_n with metric g, where a_{ji} are components of a symmetric tensor field of type $(0,2)$ in M_n. We call this metric the synectic metric. The synectic metric ${}^S g={}^C g+{}^V a$ has components [3]

$${}^s g = ({}^s\tilde{g}_{CB}) = \begin{pmatrix} a_{ji}+\partial g_{ji} & g_{ji}\\ g_{ji} & 0\end{pmatrix} \qquad (2)$$

where $\partial g_{ji} = x^{\bar{s}}\partial_s g_{ji}$.

The metric connection $\overline{\nabla}$ has components $\overline{\Gamma}^N_{AB}$ such that

$$\overline{\Gamma}^h_{ji}=\Gamma^h_{ji},\ \overline{\Gamma}^h_{j\bar{i}}=0,\ \overline{\Gamma}^h_{\bar{j}\bar{i}}=0,\ \overline{\Gamma}^h_{\bar{j}i}=0,\ \overline{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}}=0$$

$$\overline{\Gamma}^{\bar{h}}_{ji}=\partial\Gamma^h_{ji}-y^k K^h_{kji},\ \overline{\Gamma}^{\bar{h}}_{j\bar{i}}=\Gamma^h_{ji},\ \overline{\Gamma}^{\bar{h}}_{\bar{j}i}=\Gamma^h_{ji}$$

with respect to the induced coordinates in $T(M_n)$, where Γ^k_{ij} are components of ∇ in M_n [4].

2. Levi-Civita connection of ${}^S g$

Components of the Riemannian connection determined by the metric ${}^S g$ are given by

$$^{S}\Gamma_{JI}^{K}=\frac{1}{2}\widetilde{g}^{KM}\left(\partial_{J}\,{}^{S}g_{MI}+\partial_{I}\,{}^{S}g_{JM}-\partial_{M}\,{}^{S}g_{JI}\right), \tag{3}$$

where $\widetilde{g}^{KM}$ are the contravariant components of the metric ^{S}g with respect to the induced coordinates in $T(M_n)$ and

$$\widetilde{g}^{CB}=\begin{pmatrix}0 & g^{ij}\\ g^{ij} & x^{\bar{s}}\partial_{s}g^{ij}-a^{ij}_{\cdot\cdot}\end{pmatrix},\quad a^{tj}_{\cdot\cdot}=g^{it}a_{is}g^{sj}, \tag{4}$$

where g^{ij} denote the contravariant components of g in M_n [4], i.e.,

$$^{S}g_{IM}\widetilde{g}^{MJ}=\delta_{I}^{J}=\begin{cases}0, I\neq J\\ 1, I=J.\end{cases} \tag{5}$$

Then, taking account of (2) and (4), we have

$$^{s}\Gamma_{ij}^{k}=\Gamma_{ij}^{k}\ ;\ {}^{s}\Gamma_{\bar{i}j}^{k}={}^{s}\Gamma_{i\bar{j}}^{k}={}^{s}\Gamma_{\bar{i}\bar{j}}^{k}={}^{s}\Gamma_{\bar{i}\bar{j}}^{\bar{k}}=0\,,$$

$$^{s}\Gamma_{\bar{i}j}^{\bar{k}}=\Gamma_{ij}^{k}\ ;\ \ {}^{s}\Gamma_{i\bar{j}}^{\bar{k}}=\Gamma_{ij}^{k}\ ;\ \ {}^{s}\Gamma_{ij}^{\bar{k}}=x^{\bar{t}}\partial_{t}\Gamma_{ij}^{k}+H_{ij}^{k} \tag{6}$$

with respect to the induced coordinates in $T(M_n)$, Γ_{ij}^{k} being Christoffel symbols constructed with g_{ij}. Here $H_{ij}^{k}=\frac{1}{2}g^{ks}\left(\nabla_{i}a_{sj}+\nabla_{j}a_{is}-\nabla_{s}a_{ij}\right)$ is a tensor of type (1,2) and $\nabla_{k}a_{ij}=\partial_{k}a_{ij}-\Gamma_{ki}^{l}a_{lj}-\Gamma_{kj}^{l}a_{il}$.

The vertical lifts ^{V}H of $H\in \mathrm{T}\,^{1}_{2}(M_n)$ has components $^{V}H_{ji}^{\bar{k}}=H_{ji}^{k}$, all the others being zero with respect to the induced coordinates in $T(M_n)$.

From (6) we hence have

Remark 1. If $\nabla a=0$, then $^{S}\Gamma={}^{C}\Gamma$, where $^{S}\Gamma$ is Riemannian connection of ^{C}g [4].

Remark 2. If $a_{ji}=g_{ji}$, then $^{S}\Gamma={}^{C}\Gamma$.

Thus we have

Theorem 3. $^{S}\Gamma={}^{C}\Gamma+{}^{V}H$, where ^{V}H is vertical lift of $H\in \mathrm{T}\,^{1}_{2}(M_n)$.

3. Geodesics in $T(M_n)$ with ^{S}g

Let $\widetilde{C}:[0,1]\to T(M_n)$ be a curve in $T(M_n)$ and suppose that $\widetilde{C}$ is expressed locally by $x^{A}=x^{A}(t)$, i.e., $x^{h}=x^{h}(t)$, $x^{\bar{h}}=x^{\bar{h}}(t)=y^{h}(t)$ with respect to the induced coordinates (x^{h},y^{h}) in $\pi^{-1}(U)\subset T(M_n)$, *t* being a parameter. Then the curve $C=\pi\circ\widetilde{C}$ in M_n is called the projection of the curve $\widetilde{C}$ and denoted by

$\pi\widetilde{C}$ which is expressed locally by $x^h = x^h(t)$. The the curve $\widetilde{C}$ having the local expression $x^h = x^h(t)$, $y^h = dx^h/dt$ in $T(M_n)$ is called the natural lift of the curve C and denoted by C^* $[4, p.57]$.

The geodesics of the connection ${}^S\nabla$ is given by differential equations

$$\frac{d^2x^A}{dt^2} + {}^S\Gamma^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{7}$$

with respect to the induced coordinates $\left(x^h, x^{\bar{h}}\right)$, where t is an affine parameter of $\widetilde{C}$. By means of (6), (7) reduces to

$$\begin{cases} (a) \quad \dfrac{d^2x^h}{dt^2} + \Gamma^h_{ji}\dfrac{dx^j}{dt}\dfrac{dx^i}{dt} = 0 \\ (b) \, \dfrac{d^2y^h}{dt^2} + (\partial_k\Gamma^h_{ji})y^k\dfrac{dx^j}{dt}\dfrac{dx^i}{dt} + 2\Gamma^h_{ji}\dfrac{dx^j}{dt}\dfrac{dx^i}{dt} + H^h_{ji}\dfrac{dx^j}{dt}\dfrac{dx^i}{dt} = 0, \end{cases} \tag{8}$$

where Γ^h_{ji} denote components of ∇ in M_n.

Those we have

Theorem 4. Let $\widetilde{C}$ be a curve in $T(M_n)$ and locally expressed by $x^h = x^h(t)$, $y^h = y^h(t)$ with respect to the induced coordinates $\left(x^h, x^{\bar{h}}\right)$ in $T(M_n)$. The curve $\widetilde{C}$ is a geodesics of ${}^S g$, if it satisfies the equations (8).

We transform (8, (b)) as follows:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{dy^h}{dt} + \Gamma^h_{ji}\frac{dx^j}{dt}y^i\right) + \Gamma^h_{ka}\frac{dx^k}{dt}\left(\frac{dy^a}{dt} + \Gamma^a_{ji}\frac{dx^j}{dt}y^i\right) + \\ &+ \left(\partial_k\Gamma^h_{ji} - \partial_j\Gamma^h_{ki} + \Gamma^h_{ka}\Gamma^a_{ji} - \Gamma^h_{ja}\Gamma^a_{ki}\right)y^k\frac{dx^j}{dt}\frac{dx^i}{dt} + H^k_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = 0. \end{aligned} \tag{9}$$

If we put

$$\frac{\delta y^h}{dt} = \frac{dy^h}{dt} + \Gamma^h_{ji}\frac{dx^j}{dt}y^i$$

then (9) may be written as follows:

$$\frac{\delta^2 y^h}{dt^2} + \left(R^h_{kji}y^k + H^h_{ji}\right)\frac{dx^j}{dt}\frac{dx^i}{dt} = 0, \tag{10}$$

R^h_{ji} denoting the components of the curvature tensor R of ∇, which shows that the vector field $y^h(t)$ in M_n defined along $C = \pi\widetilde{C}$ is H-Jacobi field along C, where C is a geodesic in M_n, because of (8, (a)). In particular, if $H = 0$, we have Jacobi vector field along C.

Hence we have the theorem

Theorem 5. Let $\widetilde{C}$ be a geodesic in $T(M_n)$ with respect to the lift ${}^S\nabla$ of an affin connection ∇ in M_n to $T(M_n)$ and locally expressed by $x^h = x^h(t)$, $y^h = y^h(t)$ relative to the induced coordinates. Then the projection $C = \pi\widetilde{C}$ is a geodesic in M_n with respect to ∇ and the vector field $y^t(t)$ defined along *C* is a *H*- Jacobi field along geodesic *C*.

As a direct consequence of (8, (a)) and (10) with $y^h = dx^h/dt$ such that $\frac{\delta(dx^h/dt)}{dt} = 0$, we have

Theorem 6. Let $\widetilde{C}$ be a geodesic in $T(M_n)$ with respect to an affine connection ∇. The its natural lift C^* is a geodesic in $T(M_n)$ with the metric ${}^S g$.

A differentiable manifold with affine connection is said to be complete if, along an arbitrary geodesic *C*, there is a point *P* corresponding to an arbitrarily given value of affine parameter measured from a point of *C* . Thus we have this final theorem

Theorem 7. If M_n is complete with respect to an affine connection ∇, then $T(M_n)$ is complete with respect to ${}^S\nabla$, an vice versa.

References

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Sinektik metrikaların geodeziyası

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XÜLASƏ

Məqalədə sinektik ${}^{S}g={}^{C}g+{}^{V}a$ metrikasının Levi-Civita əlaqəsinə nəzərən Riman çoxobrazlısının $T(M_n)$ toxunan dəstəsinin geodeziyası öyrənilir. Burada ${}^{C}g$ -Riman metrikasının tam lifti, ${}^{V}a$ -simmetrik tenzor sahəsinin şaquli liftidir.

Açar sözlər: sinektik metrika, geodeziya, toxunan dəstə, Riman metrikası.

Геодезия синектических метрик

Мелек Арас

РЕЗЮМЕ

Основным объектом исследования статьи является геодезия касающихся пучков $T(M_n)$ Риманова многообразия относительно связи Levi-Civita синектической метрики ${}^{S}g={}^{C}g+{}^{V}a$, где ${}^{C}g$ - полный лифт Римановой метрики, ${}^{V}a$ - вертикальный лифт симметрического тензорного поля.

Ключевые слова: синектическая метрика, геодезия, касающийся пучок, Риманова метрика.