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BREIF PAPER

A NON-LINEAR GROUND STATE PROBLEM*

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Abstract. In the work the ground state problem is considered. Theorem on the largest value of the parameter involved by the functional is proved.

Keywords: ground state, symmetrization, compactness, Euler-Lagrange equation.

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1. Introduction

Let $n \geq 3$. For $\alpha > 0$ we consider the functional

$$\varepsilon_\alpha[u] = \int_{R^n} |\nabla u|^2 dx - \frac{\alpha}{2} \iint_{R^n \times R^n} \frac{u|(x)|^2 |u(y)|^2}{|x-y|^2} dxdy$$

and the ground state energy

$$E(\alpha) := \inf\{\varepsilon_\alpha[u] : u \in H^1(R^n),\ \|u\| = 1\},$$

where $\|u\| = \|u\|_2$ denotes the L^2–norm of u. Of course, $E(\alpha)$ is non-increasing with respect to α and, replacing u by $l^{n/2}u(lx)$, one easily finds that either $E(\alpha) = 0$ or $E(\alpha) = -\infty$. We are interested in the largest value of α such that $E(\alpha) = 0$, that is

$$\alpha_0 := \sup\{\alpha > 0 : E(\alpha) \geq 0\} =$$

$$= \inf\left\{ \frac{\int_{R^n} |\nabla u|^2 dx}{\frac{1}{2} \iint_{R^n \times R^n} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^2} dxdy} : u \in H^1(R^n), \|u\| = 1 \right\}.$$

Using standard compactness and symmetrization methods (see [2] and also [3]) one proves

Lemma 1. The infimum

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$$\inf\left\{\frac{\int\limits_{R^n}|\nabla u|^2\,dx}{\frac{1}{2}\iint_{R^n\times R^n}\frac{|u(x)|^2|u(y)|^2}{|x-y|^2}dxdy}:u\in H^1(R^n),\|u\|=1\right\}$$

is strictly positive and there exists a symmetric decreasing $\psi\in H^1(R^1)$ with $\|\psi\|=1$, which minimizes the ratio.

Hence we infer that $\varepsilon_{\alpha_0}|u|\geq 0$ for all u with $\|u\|=1$ and, if ψ denotes the function from the lemma,

$$\varepsilon_{\alpha_0}[\psi]=\int_{R^n}|\nabla\psi|^2\,dx-\frac{\alpha_0}{2}\iint_{R^n\times R^n}\frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|^2}dxdy=0\,.$$

Hence ψ is a minimizer of ε_{α_0} and consequently a solution of the Euler-Lagrange equation

$$-\Delta\psi-\alpha_0\int_{R^n}\frac{|\psi(y)|^2}{|x-y|^2}dy\psi=\lambda\psi$$

for some constant $\lambda<0$. Integrating against ψ we find that

$$\lambda=-\frac{\alpha_0}{2}\iint_{R^n\times R^n}\frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|^2}dxdy. \tag{1}$$

Hence $\psi_0(x):=\alpha_0^{1/2}|\lambda|^{-n/4}\psi(x/\sqrt{|\lambda|})$ satisfies

$$-\Delta\psi_0-\int_{R^n}\frac{|\psi_0(y)|^2}{|x-y|^2}dy\psi_0=-\psi_0 \tag{2}$$

and

$$\int_{R^n}|\nabla\psi_0|^2\,dx=\frac{1}{2}\iint_{R^n\times R^n}\frac{|\psi_0(x)|^2|\psi_0(y)|^2}{|x-y|^2}dxdy=\int_{R^n}|\psi_0|^2\,dx.$$

This function ψ_0 has the following variational characterization.

Theorem 1. One has

$$\int_{R^n}|\nabla\psi_0|^2\,dx=\min\left\{\int_{R^n}|\nabla u|^2\,dx:\frac{1}{2}\iint_{R^n\times R^n}\frac{|u(x)|^2|u(y)|^2}{|x-y|^2}dxdy=\|u\|^2\right\}.$$

Proof. Multiplication by constants and scaling shows that the minimum on the right side coincides with

$$\min\left\{\frac{\int\limits_{R^n}|\nabla u|^2\,dx}{\frac{1}{2}\iint_{R^n\times R^n}\frac{|u(x)|^2|u(y)|^2}{|x-y|^2}\,dxdy}: u\in H^1(R^n),\ \|u\|=1\right\}.$$

Hence by the previous lemma, it coincides with

$$\alpha_0=\frac{\int\limits_{R^n}|\nabla\psi|^2\,dx}{\frac{1}{2}\iint_{R^n\times R^n}\frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|^2}\,dxdy}.$$

On the other hand, by the definition of ψ_0

$$\int\limits_{R^n}|\nabla\psi_0|^2\,dx=\alpha_0|\lambda|^{-1}\int\limits_{R^n}|\nabla\psi|^2\,dx\,.$$

Using the value of λ from (1) we obtain the claim.

Finally, we remark that the technique from [2] might allow one to prove that the minimizer of ε_{α_0} is unique up to translations. We do not know whether this question has been investigated in arbitrary dimension. The fact that the Euler-Lagrange equation (0.2) has a unique positive, radial solution vanishing at infinity has been proved in [1] for the case $n=4$.

References

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Qeyri-xətti sıfır vəziyyət problemi

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XÜLASƏ

Məqalədə sıfır vəziyyət probleminə baxılır. Baxılan funksionala daxil olan parametrin ən böyük qiyməti haqda teorem isbat olunmuşdur.

Açar sözlər: sıfır vəziyyət, simmetrikləşdirmə, kompaktlılıq, Eyler-Laqranj tənliyi.

Нелинейная задача нулевого состояния

Ш.М. Насибов

РЕЗЮМЕ

В работе рассматривается нелинейная задача о нулевом состоянии. Доказывается теорема о наибольшем значении параметра включенного в рассмотренный функционал.

Ключевые слова: нулевое состояние, симметризация, компактность, уравнение Ейлера-Лагранжа.