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FIXED POINT THEOREM FOR NON-SELF MAPPINGS SATISFYING CONTRACTION CONDITION OF INTEGRAL TYPE IN METRICALLY CONVEX SPACES

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Abstract: We study contraction condition of integral type on non-self mappings in metrically convex metric spaces and prove a fixed point theorem for single valued non-self maps. The results generalizing and unifying fixed point theorems due to Banach [3], Branciari [4], Ciric [6], Rhoades [10] and others.

Keywords: Fixed Point, Integral type condition, Metric convexity.

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1. Introduction

In 1922, the first fundamental theorem on fixed points for contractive-type mappings was established by Banach [3] and this result is known as Banach Contraction Principle. Here for the sake of completeness, we state the result due to Banach [3] which runs as follows:

Theorem 1.1. Let $(X,\ d)$be a complete metric space, $c \in\ ]0,\ 1[$ and let $f : X \to X$ be a mapping such that for each $x, y\ \in X$,

$$d\ (fx, fy) \leq c\ d\ (x, y) \qquad (1)$$

then f has a unique fixed point $a \in X$ such that for each $x\ \in X$, $\lim_{n\to\infty} f^n x = a$.

After this classical result, there exist numerous fixed point theorems for self mappings in metric spaces and Banach spaces. However, practically speaking there do exist many situations when mappings under examination is not always a self map. So, fixed point theorems for non-self mappings are worth investigating. In this direction, Assad and Kirk [1] established a wonderful result. Since then there have been many theorems dealing with non-self mappings satisfying various types of contractive inequalities. The recent literature witness various extensions and generalizations of this theorem which includes Assad [2], Imdad et al. [7], Khan and Imdad [9], Khan [8] and others.

In 2002, Branciari [4] coined a different type of contraction condition known as contraction condition of integral type and proved a result which is as follows:

Theorem 1.2. Let (X, d) be a complete metric space, $c \in [0,1), f : X \to X$ be a mapping such that,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \int_0^{d(x,y)} \varphi(t)dt, \tag{2}$$

for each $x, y \in X$ where $\varphi: R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that, for each $\epsilon > 0, \int_0^{\epsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

The aim of this paper is to analyze the existence and uniqueness of fixed points for non-self mappings T defined on a complete metrically convex space (X, d) satisfying a contractive condition of integral type, which either partially or completely generalize the results due to Banach [3], Branciari [4], Ciric [6], Rhoades [10] and others.

Before proving the results, we collect the following definitions for further discussion.

Definition 1.1. Let (X, d) be a metric space and K be a nonempty subset of a metric space X. Let a mapping
$T: K \to X$ is said to be generalized contraction condition on K if for each $x, y \in K$,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{m(x,y)} \varphi(t)dt, \quad c \in [0,1) \tag{3}$$

where $m(x,y) = \{d(x,y), \mathrm{d(x,Tx)}, \mathrm{d(y,Ty)}, \frac{\mathrm{d(x,Ty)} + \mathrm{d(y,Tx)}}{2}\}$ and $\varphi: R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that, for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0 \ . \tag{4}$$

Definition 1.2. ([1]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that $d(x, z) + d\,(z, y) = d(x, y)$.

2. Results

The result of this paper runs as follows.

Theorem 2.1. Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T: K \rightarrow X$ be a mapping satisfying generalized contraction condition and for each $x \in \partial K, Tx \in K$. Then T has a unique fixed point $x \in K$ such that, for each $x \in K$, $\lim_{n\to\infty} T^n x = x$.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = Tx_0$. If $y_1 \in K$, set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \partial K$ so that

$$d(x_0,\ x_1) + d(x_1,\ y_1) = d(x_0,\ y_1).$$

If $y_2 \in K$, then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \partial K$ so that

$$d(x_1,\ x_2) + d(x_2,\ y_2) = d(x_1,\ y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} = Tx_n$,
(ii) $y_n = x_n$ if $y_n \in K$,
(iii) If $x_n \in \partial K$, then

$$d(x_{n-1},\ x_n) + d(x_n,\ y_n) = d(x_{n-1},\ y_n),$$

where $y_n \notin K$.

Here, one obtains two types of sets we denote as follows:

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that if $x_n \in Q$ then x_{n-1} and $x_{n+1} \in P$. We wish to estimate $d(x_{n\,,}\ x_{n+1})$. Now, we distinguish the following three cases.

Case 1. If x_n and $x_{n+1} \in P$, then

$$\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt = \int_0^{d(Tx_{n-1},\ Tx_n)} \varphi(t)dt \leq$$
$$c \int_0^{m(x_{n-1},\ x_n)} \varphi(t)dt. \qquad (5)$$

Since

$$d\,(Tx_{n-1}, Tx_n)$$
$$\leq \max\left\{d(x_{n-1},\ x_n), d(x_{n-1},\ Tx_{n-1}), d(x_n,\ Tx_n), \frac{d(x_{n-1},\ Tx_n) + d(x_n,\ Tx_{n-1})}{2}\right\}$$

$$\leq \max\left\{d(x_{n-1},\ x_n), \quad d(x_n,\ x_{n+1}), \quad \frac{d(x_{n-1},\ x_{n+1})}{2}\right\}.$$

But

$$\frac{d(x_{n-1},\ x_{n+1})}{2} \leq \frac{d(x_{n-1},\ x_n) + d(x_n,\ x_{n+1})}{2}$$
$$\leq \max\,\{d(x_{n-1},\ x_n), d(x_n,\ x_{n+1})\}.$$

Therefore

$$d(x_n,\ x_{n+1}) = d\,(Tx_{n-1}, Tx_n) \leq \max\{d(x_{n-1},\ x_n), d(x_n,\ x_{n+1})\}. \quad (6)$$

If we suppose that $d(x_{n-1},\ x_n) < d(x_n,\ x_{n+1})$, then we obtain

$$d(x_n, x_{n+1}) = d\,(Tx_{n-1}, Tx_n) \leq d(x_n, x_{n+1})$$

which is a contradiction. Therefore from equation (6), we obtain

$$d(x_n, x_{n+1}) = d\,(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n).$$

Hence

$$\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt \leq c\int_0^{m(x_{n-1}, x_n)} \varphi(t)dt. \quad (7)$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n,\ x_{n+1}) + d(x_{n+1},\ y_{n+1}) = d(x_n,\ y_{n+1}),$$

which in turn yields

$$d(x_n, x_{n+1}) \leq d(x_n,\ y_{n+1}).$$

Now, proceeding as in case 1, we have

$$\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt \leq c\int_0^{m(x_{n-1}, x_n)} \varphi(t)dt. \quad (8)$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n it follows that

$$d(x_n, x_{n+1}) \leq \max\{ d(x_{n-1}, x_{n+1}), d(y_n, x_{n+1}) \}.$$

If $d(x_{n-1}, x_{n+1}) \leq d(y_n, x_{n+1})$ then proceeding as in case 1, we have

$$\int_0^{d(x_n, x_{n+1})} \varphi(t)dt \leq c \int_0^{m(x_{n-1}, x_n)} \varphi(t)dt.$$

Otherwise if $d(x_{n-1}, x_{n+1}) \geq d(y_n, x_{n+1})$, then we have

$$\int_0^{d(x_n, x_{n+1})} \varphi(t)dt$$

$$\leq \int_0^{d(x_{n-1}, x_{n+1})} \varphi(t)dt$$

$$= \int_0^{d(Tx_{n-2}, Tx_n)} \varphi(t)dt \leq c \int_0^{m(x_{n-2}, x_n)} \varphi(t)dt. \qquad (9)$$

Here

$$d(x_n, x_{n+1}) \leq \max\left\{d(x_{n-2}, x_n), d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), \frac{d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})}{2}\right\}$$

$$\leq \max\left\{d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{2}\right\}.$$

Notice that

$$d(x_{n-2}, x_n) \leq d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \leq \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

Here, if

$$d(x_{n-2}, x_{n-1}) \leq d(x_{n-1}, x_n) \ then \ d(x_{n-2}, x_n) \leq d(x_{n-1}, x_n).$$

Otherwise, if

$$d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}) \ then \ d(x_{n-2}, x_n) \leq d(x_{n-2}, x_{n-1}).$$

Therefore, we obtain

$$d(x_n, x_{n+1}) \leq \max\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{2}\right\}$$

which in turn yields

$$d(x_n,\ x_{n+1}) \le \begin{cases} c\, d(x_{n-1},\ x_n)\ if\ d(x_{n-1},\ x_n) \ge d(x_{n-2},\ x_{n-1}) \\ c\, d(x_{n-2},\ x_{n-1})\ if\ d(x_{n-1},\ x_n) \le d(x_{n-2},\ x_{n-1}). \end{cases}$$

Thus in all the cases, we have

$$d(x_n,\ x_{n+1}) \le c \max\{d(x_{n-1},\ x_n), d(x_{n-2},\ x_{n-1})\}. \tag{10}$$

It can be easily shown by induction that for $n > 1$, we have

$$d(x_n,\ x_{n+1}) \le c \max\{d(x_0,\ x_1), d(x_1,\ x_2)\}. \tag{11}$$

Thus

$$\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt \le c \int_0^{\max\{\{d(x_0,\ x_1), d(x_1,\ x_2)\}\}} \varphi(t)dt$$

which implies that

$$\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt \le c\, max \left\{\int_0^{d(x_0,\ x_1)} \varphi(t)dt, \int_0^{d(x_1,\ x_2)} \varphi(t)dt\right\}. \tag{12}$$

It follows that the sequence $\{d(x_n,\ x_{n+1})\}$ is monotonically decreasing. Hence $\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt \to 0$ as $n \to \infty$. From equation (4) it implies that

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{13}$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let on contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{n_k},\ x_{m_k}) \ge \epsilon$.

Here, we proceed on the lines of Rhoades [10], it can be shown that the sequence $\{x_n\}$ is Cauchy and converges to a point say x. From equation (3) we have

$$\int_0^{d\,(Tx,\ x_{n+1})} \varphi(t)dt \le c \int_0^{m(x,x_n)} \varphi(t)dt \le c\, max \left\{\int_0^{d(x,\ x_n)} \varphi(t)dt, \int_0^{d(x,\ Tx)} \varphi(t)dt, \int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt, \int_0^{d(x,\ x_{n+1})} \varphi(t)dt, \int_0^{d(x_n,\ Tx)} \varphi(t)dt, \right\}. \tag{14}$$

On letting $k \to \infty$, in equation (14) then we have,

$$\int_0^{d\,(Tx,\ x)} \varphi(t)dt \ \le c \int_0^{d\,(Tx,\ x)} \varphi(t)dt$$

which implies that

$$\int_0^{d\,(Tx,\ x)} \varphi(t)dt \ = 0, \tag{15}$$

which from equation (15), implies that $d\ (Tx, x) = 0$, this implies that $Tx = x$. This shows that x is a fixed point T.

To prove that the uniqueness of fixed points. Let us suppose that x_1 and x_2 are two fixed points of T, then

$$\int_0^{d(x_1,\ x_2)} \varphi(t)dt = \int_0^{d(Tx_1,\ Tx_2)} \varphi(t)dt \le c \int_0^{m(x_1,\ x_2)} \varphi(t)dt$$
$$= c\,max\left\{\int_0^{d(x_1,\ x_2)} \varphi(t)dt\,, \qquad 0\right\} = c \int_0^{d(x_1,\ x_2)} \varphi(t)dt$$

which implies that $\int_0^{d(x_1,\ x_2)} \varphi(t)dt \ = 0$. Also imply that $d(x_1,\ \ x_2) = 0$ or $x_1 = x_2$. This shows the uniqueness of fixed point. This completes the proof.

Remark 2.1. By setting $K = X$ and $\varphi(t) = 1$ for each $t\ \ge 0$ in the Theorem 2.1, then we deduce a partial generalization of the result due to Banach [3].
Remark 2.2. By setting $K = X$ in the Theorem 2.1, then we deduce a result due to Rhoades [10].
Remark 2.3. By setting $K = X$ in the Theorem 2.1, then we deduce a fine result due to Branciari [4].

By setting $K = X$ in the Theorem 2.1, then we deduce the following corollary in the form of the result due Ciric [6].

Corollary 2.1. Let (X, d) be a complete metric space, $c\ \in [0,1[,\ T: X\ \to X$ is a mapping such that, for each $x, y\ \in X$,

$$\int_0^{d(Tx,\ Ty)} \varphi(t)dt \le c\ \int_0^{m(x,\ y)} \varphi(t)dt \tag{16}$$

where $m\ (x, y) = \max\{d(x, y), \mathrm{d(x,Tx)}, \mathrm{d(y,Ty)}, \mathrm{d}(x, \mathrm{Ty}), \mathrm{d(y,Tx)}\}$ and $\varphi: R^+ \to R^+$is a Lebesgue-integrable mapping which is summable, non-negative and such that, for each $\epsilon\ > 0, \int_0^{\epsilon} \varphi(t)dt\ > 0$. Then T has a unique fixed point $z\ \in X$ such that, for each $x\ \in X, \lim\limits_{n\to\infty} T^n x = z$.

Example 2.1. Consider $X = R$ be the set of reals equipped with natural distance and $K = \left\{\frac{1}{n}: n \in Z, |n| \ge 2\right\} \cup 0.$ Define $T: K\ \to X$ by

$$T\left(\frac{1}{n}\right)=\begin{cases}\frac{1}{n-1}, & if\ n>1, \quad n\ is\ odd\\ \frac{1}{n}, & if\ n>0, \quad n\ is\ even\\ \frac{1}{n-1}, & if\ n<0, \quad n\ is\ even\\ 0, & if\ n\to\infty.\end{cases}$$

This example shows that Theorem 2.1 is a proper extension and generalization of the earlier results due to Rhoades [10], Branciari [4] and others.

Conclusion: Theorem 2.1 generalizes the main results of Banach [3], Branciari [4], Ciric [6], Rhoades [10] and others. Moreover, we have considered the domain of our mapping is non-self rather than the self mapping. This shows a very general nature of our result in contrast to other known results in the literature. Finally, the above example gives an insight view of our result and applicable superiority over other results.

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