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THE $G_{m,n}^M$ GRAPH ON A FINITE SUBSET OF NATURAL NUMBERS

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Abstract. The undirected graph $G_{m,n}^M = (V, E)$ has the vertex set $V = \{1,2,3,\dots,n\}$ and $u, v \in V$ are adjacent if and only if $u \neq v$ and $u \cdot v$ is not divisible by m, where $m, n \in \mathbb{N}$. The connectedness, the completeness, the diameter and the Eulerian property of $G_{m,n}^M$ are explored in this paper. The average degree, the top, the gap and the balanced conditions of $G_{m,n}^M$ for various values of m are also analysed.

Keywords: Connected graph, complete graph, split graph, clique, independent set, Eulerian property, average degree, the top, the gap, balanced graph.

AMS Subject Classification (2020): 05C05, 05C40

1. Introduction

Let $m, n \in \mathbb{N}$, where $\mathbb{N}$ denote the set of all natural numbers. In this paper, we define and study an undirected simple graph $G_{m,n}^M = (V, E)$ on a finite subset of natural numbers, where the vertex set $V = \{1, 2, \dots, n\}$ and any two distinct vertices $u, v \in V$ are adjacent if and only if $m \nmid u \cdot v$. We study the connectedness, the completeness, the edge degree, the diameter and the Eulerian property of $G_{m,n}^M$. We determine the values of m such that the average degree of non regular graph $G_{m,n}^M$ is an integer. We also find the values of m such that $G_{m,n}^M$ is balanced. One can refer [1, 3] for graphs defined and studied on finite subset of natural numbers.

Throughout the paper for a vertex $i \in V$, we mean the label of the vertex $v = i$ and uv denote the usual multiplication $u \cdot v$. For terminology and notations that are not defined here, we follow [7].

2. Connectedness of $G_{m,n}^M$

We begin with some simple observations.

Observation 2.1. Let $m = 1$. Then $G_{m,n}^M$ is a null graph with n vertices.

Observation 2.2. For $1 < m \leq n$, the graph $G_{m,n}^M$ is disconnected.

We now present a structural property of the $G_{m,n}^{M}$ graph.

Theorem 2.1. Let $1 < m \leq n$ and m is a prime. Then $G_{m,n}^{M}$ is disjoint union of $K_{\mathrm{n}-\lfloor\frac{n}{m}\rfloor}$ and $\lfloor\frac{n}{m}\rfloor$ copies of K_1.

Proof. Let m be a prime, where $1 < m \leq n$. The number of multiples of m up to n is $\lfloor\frac{n}{m}\rfloor$. The vertex set V of $G_{m,n}^{M}$ can be written as the disjoint union of the sets V_1 and V_2, where
$V_1 = \{i \in V : gcd(i,m) = 1\}$ and $V_2 = \{j \in V : gcd(j,m) = m\}$. Let $i_1, i_2 \in V_1$, then $gcd(i_1,m) = 1$ and $gcd(i_2,m) = 1$, which gives $gcd(i_1 \cdot i_2, m) = 1 \Rightarrow m \nmid i_1 \cdot i_2$, thus the vertices i_1, i_2 are adjacent. Let $i_1 \in V_1$, $j_1 \in V_2$, then $gcd(i_1,m) = 1$, $gcd(j_1,m) = m \Rightarrow m | i_1 \cdot j_1$. So, the vertices i_1, j_1 are not adjacent. Hence the vertices in V_1 are not adjacent to any vertex $j \in V_2$ as $m | i_1 \cdot j$, $i_1 \in V_1$, $j \in V_2$. Again, the vertices in V_1 form a clique of size $n - \lfloor\frac{n}{m}\rfloor$. The vertices in V_2 are not adjacent to each other as $m | j \cdot k, j, k \in V_2$. And the cardinality of the set V_2 is $\lfloor\frac{n}{m}\rfloor$. Thus the graph $G_{m,n}^{M}$ is disjoint union of $K_{\mathrm{n}-\lfloor\frac{n}{m}\rfloor}$ and $\lfloor\frac{n}{m}\rfloor$ copies of K_1. The graph shown in Figure 1 illustrates the disconnectedness and the components of $G_{m,n}^{M}$ for $m < n$.

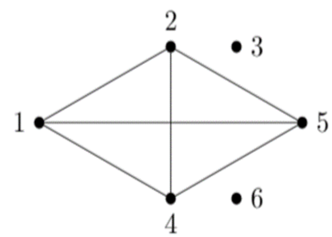

Figure 1: The graph $G_{m,n}^{M}$ for $m = 3, n = 6$.

Connectedness of $G_{m,n}^{M}$ is explained in the next theorem.

Theorem 2.2. Let $m > n$. Then the graph $G_{m,n}^{M}$ is connected.

Proof. Let $m > n$ and the vertex set $V = \{1, 2, \dots, n\}$. As $m > n \Longrightarrow m > 1 \cdot n \Longrightarrow m > 1 \cdot j$ for all $j \in \{2, 3, \dots, n\}$, which gives $m \nmid 1 \cdot j$ for all $j \in \{2, 3, \dots, n\}$. Thus the vertex $i = 1$ is adjacent to all other vertices in $G_{m,n}^{M}$, which follows that $G_{m,n}^{M}$ is connected for $m > n$.

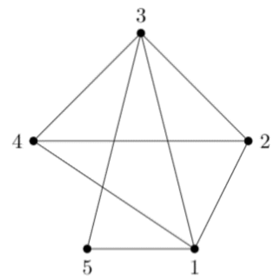

Figure 2: The graph $G_{m,n}^{M}$ for $m = 10, n = 5$.

3. Counting the Degree of a Vertex

In this section, we explain how to count the degrees of vertices of $G_{m,n}^{M}$ for natural numbers m, n. Then we explain how to calculate the total number of distinct possible degrees in $G_{m,n}^{M}$ of order n for various values of m.

We determine the degree of a vertex as follows:

- If the $gcd(i,m) = 1$, then the degree of the vertex i is $deg(i) = n - 1$.
- If the $gcd(i,m) = i > 1$ and $m = i \cdot j$, where $j > n$, then the degree of the vertex i is $n - 1$.
- If the $gcd(i,m) = i > 1$ and $m = i \cdot j$, where $j \leq n$, such that $gcd(i,j) = 1$, then the degree of the vertex i is $n - \lfloor \frac{n}{j} \rfloor - 1$.
- If the $gcd(i,m) = i > 1$ and $m = i \cdot j$ $(i \neq j, i, j \leq n)$, where $i|j$, then the degree of the vertex i is $deg(i) = n - \lfloor \frac{n}{j} \rfloor - 1$ and the degree of the vertex j is $deg(j) = n - \lfloor \frac{n}{i} \rfloor$.
- If the $gcd(i,m) = i > 1$ and $m = i \cdot i$, then the degree of the vertex i is $deg(i) = n - \lfloor \frac{n}{i} \rfloor$.
- If $gcd(i,m) = j > 1$ and $m = j \cdot k$, then the degree of the vertex i is $n - \lfloor \frac{n}{k} \rfloor - 1$.

Lemma 3.1. Let $i, j \in V$. Then the degrees of the vertices i, j are equal if $gcd(i,m) = gcd(j,m)$.

Proof. Let $gcd(i,m) = gcd(j,m) = i_1,\ i,\ j \in V,\ i_1 \in \mathbb{N}$. Then degrees of the vertices i, j are same as $m = i_1 \cdot \frac{m}{i_1}$, then the degree of the vertices i, j is

$$n - \left\lfloor \frac{n}{\frac{m}{i_1}} \right\rfloor - 1.$$

For a given value of m, the various possible degrees of $G_{m,n}^{M}$ of order n is explained. We find the pair of factors $\{i_t\ ,\ j_t\}$, for $i, j = 1, 2, \ldots, t$ of m such that

$$m = i_1 \cdot j_1 = i_2 \cdot j_2 = \dots = i_t \cdot j_t$$

and each of these factors of m, that is, $i_1, i_2, \dots, i_t, j_1, j_2, \dots, j_t$ are less than or equal to n. Then the possible degrees of $G^M_{m,n}$ are $\mathcal{A} = \left\{n-1, n-\lfloor\frac{n}{j_1}\rfloor - 1 \text{ or } n - \left\lfloor\frac{n}{j_1}\right\rfloor, n - \left\lfloor\frac{n}{i_1}\right\rfloor - 1 \text{ or } n - \left\lfloor\frac{n}{i_1}\right\rfloor, \dots, n - \left\lfloor\frac{n}{j_t}\right\rfloor - 1 \text{ or } n - \left\lfloor\frac{n}{j_t}\right\rfloor, n - \left\lfloor\frac{n}{i_t}\right\rfloor - 1 \text{ or } n - \left\lfloor\frac{n}{i_t}\right\rfloor\right\}$. The cardinality of the set $\mathcal{A}$ gives the number of distinct possible degrees of $G^M_{m,n}$.

It is natural to count the minimum number of vertices of degree $n-1$. In fact, the number of vertices of degree $n-1$ allow us to know the minimum number of integers in $\{1, 2, \dots, n\}$ that are co-prime to m. Let us assume that $p_1, p_2, \dots, p_k$ be the k distinct primes present in the prime factorization of m, that is, $m = {p_1}^{n_1}{p_2}^{n_2} \dots {p_k}^{n_k}$ and $\mathcal{B}$ be the number of the vertices in V whose labels are relatively prime to m. Let P_l be the property that an integer is divisible by prime p_l, for $l = 1, 2, \dots, k$. Let $A\_l = \{x : x \in V$ and x has property $P\_l\}$. Then $A_l \cap A_j$ is a subset of V that have both property P_l and P_j. Similarly, $A_l \cap A_j \cap A_o$ is a subset of V that have the property P_l, P_j and P_o and so on. Thus by using inclusion-exclusion principle, we have $\mathcal{B} = n - \sum|A_i| + \sum|A_l \cap A_j| - \sum|A_l \cap A_j \cap A_o| + \dots + (-1)^k|A_1 \cap A_2 \cap \dots \cap A_k|$, where the first sum is over all 1-combinations j of $\{1,2,\dots,k\}$, the second sum is over all 2-combinations $\{l, j\}$ of $\{1,2,\dots,k\}$ and so on.

Example 3.1. Let $n = 21$ and $m = 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6$. Then the possible degrees of $G^M_{m,n}$ are $\mathcal{A} = \{n-1, n-\lfloor\frac{n}{18}\rfloor - 1, n - \lfloor\frac{n}{2}\rfloor, n - \lfloor\frac{n}{12}\rfloor - 1, n - \lfloor\frac{n}{3}\rfloor, n - \lfloor\frac{n}{9}\rfloor - 1, n - \lfloor\frac{n}{4}\rfloor - 1, n - \lfloor\frac{n}{6}\rfloor\} = \{n-1, n-2, n-10, n-2, n-7, n-3, n-6, n-6\} = \{n-1, n-2, n-3, n-6, n-7, n-10\}$. Thus, the distinct possible degrees of $G^M_{36,21}$ is the cardinality of $\mathcal{A}$, which is 6. Again, the number of vertices in V whose labels are co-prime to m is $\mathcal{B} = n - (\lfloor\frac{n}{2}\rfloor + \lfloor\frac{n}{3}\rfloor) + \lfloor\frac{n}{6}\rfloor = 7$.

Now we find the degree of the vertices. As $gcd(1,36) = gcd(5,36) = gcd(7,36) = gcd(11,36) = gcd(13,36) = gcd(17,36) = gcd(19,36) = 1$, the degree of the vertices $\{1, 5, 7, 11, 13, 17, 19\}$ are $n - 1 = 20$. Again $gcd(2,36) = gcd(10,36) = gcd(14,36) = 2$, thus the degree of the vertices $\{2, 10, 14\}$ are equal. The degree of the vertex labeled as 2 is $deg(2) = n - \lfloor\frac{n}{18}\rfloor - 1 = n - 2 = 19$. The degree of the vertex $v = 3$ is $deg(3) = n - \lfloor\frac{n}{12}\rfloor - 1 = n - 2 = 19$. As $gcd(3,36) = gcd(15,36) = gcd(21,36) = 3$, so the degree of the vertices $15, 21$ are $n - 2 = 19$. Similarly, it can be seen that the degrees of the vertices $4, 8, 16, 20$ are $n - 3 = 18$. The degree of the vertex $u = 6$ is $deg(6) = n - \lfloor\frac{n}{6}\rfloor = n - 3 = 18$. The vertex $v = 9$ is of degree $n - \lfloor\frac{n}{4}\rfloor - 1 = 15$. The degree of the vertex $v = 12$ is $deg(12) = n - \lfloor\frac{n}{3}\rfloor = n - 7 = 14$ and the degree of the vertex $v = 18$ is $deg(18) = n - \lfloor\frac{n}{2}\rfloor = n - 10 = 11$. Hence the various possible degrees of the vertices are $20, 19, 18, 15, 14, 11$.

It is known that the number of divisors of x is denoted by $\sigma_0(x)$.
Let D be the number of distinct possible degrees of $G^M_{m,n}$. We study the relation between D and $\sigma_0(m)$.
For $1 < m \leq n$, let us define a binary relation ρ on V as follows:
For $a, b \in V$, $a\rho\, b \Leftrightarrow gcd(a, m) = gcd(b, m)$.
Clearly ρ is an equivalence relation on V. Thus ρ partition the vertex set V into equivalence classes. And the number of equivalence classes is equal to the number of distinct factors of m that are less than or equal to n.

Theorem 3.2. For $1 < m \leq n$, $D = \sigma_0(m)$.

Proof. Let $1 < m \leq n$ and $f_1, f_2, \ldots, f_t$ are the factors of m, then clearly the factors of m are also less than or equal to n. It is clear that for any vertex $u \in V$, $gcd(u, m) = f_i$, where f_i,
is a factor of m, for $i = 1,2, \ldots, t$. Thus, the vertex set V can be partitioned into t disjoint subsets such as $V_1, V_2, \ldots, V_t$, where $t = \sigma_0(m)$ and each subset contain vertices of V which have the same gcd with m.
We claim that the number of distinct possible degrees D is equal to $\sigma_0(m)$. Let, if possible, $D > \sigma_0(m)$. Then there will a vertex $w \in V$ such that $gcd(w, m) = m_1$, where m_1 is not a factor of m, which is absurd. Again, consider the case that $D < \sigma_0(m)$.
Then there will be at least two partitions of the subsets of V such that all the vertices in both the partition bear the same degree of the vertices. Let the two partitions of V be V_i and V_j where $gcd(v_i, m) = f_i$ and $gcd(u_j, m) = f_j$, for $v_i \in V_i$, $u_j \in V_j$ and f_i, f_j are factors of m. Then $\frac{n}{m_i} = \frac{n}{m_j}$, where $m = f_i \cdot m_i = f_j \cdot m_j$, which is not possible as $f_i \neq f_j$ and $f_i, f_j, m_i, m_j \leq n$.
Thus $D = \sigma_0(m)$.

Example 3.2. Consider $G^M_{m,n}$ where $m = 8$ and $n = 10$. Then the number of factors of m is $\sigma_0(m) = 1 + 2^0 + 4^0 + 8^0 = 4$. The vertices $1, 3, 5, 7$ are co-prime to $m = 8$, so $deg(1) = deg(3) = deg(5) = deg(7) = n - 1 = 9$. Again, the vertices $2, 6, 10$ are not adjacent to the vertices $4, 8$ implying $deg(2) = deg(6) = deg(10) = n - 3 = 7$. The vertex 4 is not adjacent to the vertices $2, 6, 8, 10$, so $deg(4) = 5$ and clearly the vertex 8 is of degree 0. Thus $G^M_{8,10}$ is a disconnected graph with $\sigma_0(8) = 4$ distinct degrees such as $9, 7, 5, 0$.

4. The Structure of $G^M_{m,n}$ for Various Values of $m > n$

For a given value of n, if $m > n$, $G^M_{m,n}$ is connected by Theorem 2.2. In this section we consider the structure of $G^M_{m,n}$ graphs where $m > n$. It is easy to see that $G^M_{m,n}$ is complete for $m > n(n-1)$ as $m > n(n-1) \Longrightarrow m > i \cdot j$, which gives $m \nmid i \cdot j$ for all $i, j \in V$.

We study the graph $G^M_{m,n}$, where m takes the value as mentioned below.

- a prime,
- a multiple of a prime,
- a square of a prime $\lfloor \frac{n}{2} \rfloor < p \leq n$,
- a square of a prime $1 < p \leq \lfloor \frac{n}{2} \rfloor$,
- product of $i, j \in V$ such that both $i, j \leq \lfloor \frac{n}{2} \rfloor$,
- product of $i, j \in V$ such that both $\lfloor \frac{n}{2} \rfloor < i, j \leq n$,
- product of $i, j \in V$ such that $i < \lfloor \frac{n}{2} \rfloor$ and $j > \lfloor \frac{n}{2} \rfloor$.

Theorem 4.1. Let $n < m \leq n(n-1)$. The graph $G^M_{m,n}$ is complete if

(i) m is a prime;

(ii) m is a multiple of a prime $p > n$;

(iii) $\frac{m}{i} > n$, for $i \leq n$ is a factor of m;

(iv) $m = p^2$, where p is a prime and $\lfloor \frac{n}{2} \rfloor < p \leq n$.

Proof. **(i)** Let m be a prime. Then, for all $i, j \in V$ $m \nmid i \cdot j$, which implies $deg(i) = deg(j) = n - 1$. Thus, the graph $G^M_{m,n}$ is complete.

(ii) Let $m = m_1 \cdot p$, where $m_1 \in \mathbb{N}$ and $p > n$ be a prime. Then $p \nmid i \cdot j \Longrightarrow m \nmid i \cdot j$ for all $i, j \in V$, which implies $G^M_{m,n}$ is complete.

(iii) Let $i < n$ be a factor of m and $\frac{m}{i} > n$, then clearly $\lfloor {}^{n}/_{\frac{m}{i}} \rfloor = 0$, which implies $deg(i) = n - 1$. Again, if $j \in V$ such that j is not a factor of m, then $gcd(j, m) = 1$ which gives the degree of the vertex j is $n - 1$. Hence $G^M_{m,n}$ is complete.

(iv) Let $m = p^2$, where p is a prime and $\lfloor \frac{n}{2} \rfloor < p \leq n$, then the $gcd(i, m) = 1$ for all $i (\neq p) \in V$ implying $deg(i) = n - 1$ for all $i \in V$. It is clear that $\lfloor \frac{n}{p} \rfloor = 1$, which implies the degree of the vertex $p \in V$ is $deg(p) = n - 1$.

Theorem 4.2. The maximum degree of the graph $G^M_{m,n}$ is $n - 1$.

Proof. Clearly, $n - 1$ is the highest possible degree of the graph $G^M_{m,n}$ with n vertices. The vertex $i = 1 \in V$ is of degree $n - 1$ as $m \nmid 1 \cdot j$ for all $j \in V$.

Lemma 4.3. Let $n < m \leq n(n-1)$ and $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ is a prime. Then (n-1)/2 vertices are of degree $n - 1$, $(n-1)/2$ vertices are of degree $n - 2$ and one vertex is of degree $(n-1)/2$, if n is odd; and $(n-2)/2$ vertices are of degree $n - 1$, $n / 2$ vertices are of degree $n - 2$ and one vertex is of degree $(n-2)/2$, if n is even.

Proof. Let $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} \rfloor + 1, \dots, n\}$ is a prime, then m divides only the even multiples of p. Thus, any vertex $j \in V$ labeled as odd integer (except the vertex p) is adjacent to all the vertices as $m \nmid j \cdot k$, for all $k (\neq j) \in V$ which gives the degree of the vertex j is $n-1$. The vertices labeled as even integers are of degree $n-2$ as they are not adjacent to the vertex p (as $m | w \cdot p$, where the label of the vertex w is even) and itself. The vertex p is adjacent to the vertices labeled as odd integers (except itself) as $2p \nmid j \cdot p$, where $j \in V$ is an odd integer. The following two possibilities may arise:

Case I. Let n be odd, then $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ vertices are labeled as odd integers and $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ vertices are labeled as even integers. Hence $\frac{n+1}{2} - 1 = (n-1)/2$ vertices are of degree $n-1$, $(n-1)/2$ vertices are of degree $n-2$. The vertex $w = p$ is of degree $\frac{n+1}{2} - 1 = \frac{(n-1)}{2}$.

Case II. Let n be even, then $n/2$ vertices are labeled as odd integers and $n/2$ vertices are labeled as even integers. Thus $n/2 - 1 = (n-2)/2$ vertices are of degree $n-1$, $n/2$ vertices are of degree $n-2$ and the vertex $w = p$ is of degree $n/2 - 1 = (n-2)/2$.

Lemma 4.4. Let $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$, then $G_{m,n}^{M}$ contain $n - \lfloor \frac{n}{p} \rfloor$ vertices of degree $n-1$ and $\lfloor \frac{n}{p} \rfloor$ vertices of degree $n - \lfloor \frac{n}{p} \rfloor$.

Proof. Let $m = p^2$, where p is a prime and $p \leq \lfloor \frac{n}{2} \rfloor$. The vertex $u \in V$ is of degree $n-1$ if the label of u is not multiple of p. The number of multiple of p up to n is $\lfloor \frac{n}{p} \rfloor$. Thus $n - \lfloor \frac{n}{p} \rfloor$ vertices are of degree $n-1$. Again, let $w \in V$, such that the label of w is multiple of p, then $deg(w) = n - \lfloor \frac{n}{p} \rfloor$ as w is not adjacent to the vertices whose labels are multiples of p. Thus $\lfloor \frac{n}{p} \rfloor$ vertices are of degree $n - \lfloor \frac{n}{p} \rfloor$.

It is known that a split graph is a simple graph in which the vertices can be partitioned into a disjoint union of clique and an independent set [4, 5].

Theorem 4.5. For $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$, $G_{m,n}^{M}$ is a split graph.

Proof. Let $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 4.4, the vertex set V of $G_{m,n}^{M}$ can be partitioned into two disjoint subsets V_1, V_2 of V such that V_1 consist of the vertices of degree $n-1$ and V_2 consist of the vertices of degree $n - \lfloor \frac{n}{p} \rfloor$. Again, the vertices in V_1 form a clique of size $n - \lfloor \frac{n}{p} \rfloor$ and, on the other hand the vertices of V_2 form independent set of size $\lfloor \frac{n}{p} \rfloor$. Hence $G_{m,n}^{M}$ is a split graph.

Lemma 4.6. For $m = i \cdot j$ and i, j are the unique factors of m such that $i, j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\} \subseteq V$, then the possible degrees of the vertices in $G^M_{m,n}$ are $\{n-1,\ n-2\}$.

Proof. Let $m = i \cdot j$ such that i, j are the only factors of m, where $\left\lfloor\frac{n}{2}\right\rfloor + 1 \leq i, j \leq n$. Then the degree of the vertex i is $deg(i) = n - \lfloor\frac{n}{j}\rfloor - 1$ and the degree of the vertex j is $deg(j) = n - \lfloor\frac{n}{i}\rfloor - 1$. As $\lfloor\frac{n}{2}\rfloor + 1 \leq i, j \leq n$, so $\lfloor\frac{n}{i}\rfloor = \lfloor\frac{n}{j}\rfloor = 1$. Thus $deg(i) = deg(j) = n - 2$.
To find the degree of a vertex $u \in V$ where $u \neq i, j$, we may consider the following cases:
Case I. Let $u, v \in \{1,2,.., \lfloor\frac{n}{i}\rfloor\} \subseteq V$, then $u \cdot v < i \cdot j = m$, which implies $m \nmid u \cdot v$. Thus, the vertices u, v are adjacent.
Case II. Let $u, v \in V$ such that $u \in \{1, 2, \dots, \lfloor\frac{n}{i}\rfloor\}$, $v \in \{\lfloor\frac{n}{i}\rfloor + 1, \dots, n\}$. Then $u \cdot v < i \cdot j = m$, thus $m \nmid u \cdot v$, hence the vertices u, v are adjacent.
Case III. Let $u, v \in \{\lfloor\frac{n}{i}\rfloor + 1, \dots, n\}$. As $u \neq i, j$ and i, j are the unique factors of m, so $u \cdot v \neq i \cdot j$ for all v, which implies $m \nmid u \cdot v$, thus u, v are adjacent.
Hence from all the three cases it follows that for $u \neq i, j$, the degree of the vertex u is $n - 1$. So, the possible degrees of the vertices are $\{n-1, n-2\}$.

Theorem 4.7. For $m = i \cdot j$, where i, j are the unique factors of m such that $i, j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\} \subseteq V$, then $G^M_{m,n}$is a split graph.

Proof. Let i, j are the unique factors of m such that $m = i \cdot j$, $i, j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\} \subseteq V$. By Lemma 4.6, the vertices i, j are of degree $n - 2$ and all other vertices are of degree $n - 1$. Moreover, the vertices i, j are independent and the vertices in $V_1 = V \setminus \{i, j\}$ are of degree $n - 1$ forming a clique of size $n - 2$. Thus, the vertex set $V = V_1 \cup \{i, j\}$, where V_1 is a clique and $\{i, j\}$ is an independent set. Thus $G^M_{m,n}$ is a class of split graph.

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. In [6], R. Gera et al. defined the edge degree of an edge $\{a, b\} \in E_\Gamma$ as follows:

$$deg(\{a, b\}) = deg(a) + deg(b) - 2.$$

In this section the next few results are about the possible edge degrees and the sum of edge degree sequence of $G^M_{m,n}$ for various values of m.

Theorem 4.8. For the unique factors i, j of m such that $m = i \cdot j$, where $i, j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\} \subseteq V$, the possible edge degrees are $2n - 4, 2n - 6, 2n - 5$ and the sum of edge degree sequence is $n^3 - 7n^2 + 50n - 98$.

Proof. Let $i,\ j$ are the unique factors of m such that $m = i \cdot j$, where $i, j \in \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, n\}$. Then by Lemma 4.6, $G_{m,n}^{M}$ contain $n-2$ vertices of degree $n-1$ and 2 vertices of degree $n-2$. Thus, the degree of an edge $\{a_1, b_1\}$, where the vertices a_1, b_1 both are of degree $n-1$, is $n-1+n-1-2 = 2n-4$. Similarly, we find the degree of an edge $\{a_2, b_2\}$ is $2n-6$ if both the vertices a_2, b_2 are of degree $n-2$ and the degree of an edge $\{a_3, b_3\}$ is $2n-5$ if the vertices a_3, b_3 are of degree $n-1$ and $n-2$ respectively.
The sum of edge degree sequence is given by
$\sum_{\forall\{a,b\}\in E} deg(\{a,b\}) = \{(n-2)+(n-2)-2\} + \binom{n-2}{2}\{n-1+n-1-2\} + 2(n-2)\{n-1+n-2-2\} = n^3 - 7n^2 + 50n - 98.$

Theorem 4.9. For $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$, the possible edge degrees are $2n-4$, $2(n - \lfloor \frac{n}{p} \rfloor - 1)$, $2n - \lfloor \frac{n}{p} \rfloor - 3$ and sum of edge degree sequence is $n^3 + (n-1)(\left\lfloor \frac{n}{p} \right\rfloor - \left(\left\lfloor \frac{n}{p} \right\rfloor\right)^2) - n(3n-2)$.

Proof. Let $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$, then Lemma 4.4 asserts that there are $\lfloor \frac{n}{p} \rfloor$ vertices of degree $n - \lfloor \frac{n}{p} \rfloor$ and $n - \lfloor \frac{n}{p} \rfloor$ vertices of degree $n-1$. Thus, the possible edge degrees of $G_{m,n}^{M}$ are $2n-4$, $2(n - \lfloor \frac{n}{p} \rfloor - 1)$, $2n - \lfloor \frac{n}{p} \rfloor - 3$. And the sum of the edge degree of $G_{m,n}^{M}$ is given by

$$\sum_{\forall\{a,b\}\in E} deg(\{a,b\}) = \binom{n - \left\lfloor \frac{n}{p} \right\rfloor}{2}(n-1+n-1-2) + \binom{\left\lfloor \frac{n}{p} \right\rfloor}{2}\left(n - \left\lfloor \frac{n}{p} \right\rfloor + n - \left\lfloor \frac{n}{p} \right\rfloor - 2\right) + \lfloor \frac{n}{p} \rfloor (n - \lfloor \frac{n}{p} \rfloor)(n-1+n - \lfloor \frac{n}{p} \rfloor - 2) = n^3 + (n-1)(\lfloor \frac{n}{p} \rfloor - (\lfloor \frac{n}{p} \rfloor)^2) - n(3n-2).$$

Theorem 4.10. For $n < m \leq n(n-1)$ and $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ is a prime, the sum of edge degree sequence is $n(n-1)(n-2) - \lfloor \frac{n}{2} \rfloor (n-1)(2n-3) + \left(\lfloor \frac{n}{2} \rfloor\right)^2 (2n-5)$.

Proof. Let $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ is a prime. Applying Lemma 4.3, we find the edge degrees by considering two cases.
Case I. Let n be odd then edge degrees of $G_{m,n}^{M}$ are $2n-4$, $2n-6$, $\frac{3n-7}{2}$, $\frac{3n-9}{2}$. And the sum of edge degree sequence of $G_{m,n}^{M}$ is given by $\sum_{\forall\{a,b\}\in E}\{a,b\} = \binom{\frac{n-1}{2}}{2}(n-$

$1 + n - 1 - 2) + \binom{\frac{n-1}{2}}{2}(n - 2 + n - 2 - 2) + (\frac{n-1}{2})(n - 1 + \frac{n-1}{2} - 2) + \frac{n-1}{2}(n - 2 + \frac{n-1}{2} - 2) = \frac{1}{4}(n-1)(2n^2 - 5n - 1).$

Case II. Let n be even then edge degrees of $G_{m,n}^M$ are $2n-4$, $2n-6$, $\frac{3n-8}{2}$, $\frac{3n-10}{2}$. And the sum of edge degree sequence of $G_{m,n}^M$ is given by $\sum_{\forall\{a,b\}\in E}\{a,b\} = \binom{\frac{n-2}{2}}{2}(n - 1 + n - 1 - 2) + \binom{\frac{n}{2}}{2}(n - 2 + n - 2 - 2) + (\frac{n-2}{2}\}(n - 1 + \frac{n-1}{2} - 2) + \frac{n}{2}(n - 2 + \frac{n-2}{2} - 2) = \frac{1}{4}[2n^3 - 7n^2 + 2n].$

Theorem 4.11. The diameter of $G_{m,n}^M$ is 1, 2 or ∞.

Proof. To find the diameter of $G_{m,n}^M$ we consider the following cases.
Case I. Let $m < n$. Then the graph $G_{m,n}^M$ is disconnected. So, the diameter of $G_{m,n}^M$ is ∞.
Case II. Let $m > n(n-1)$, then the graph $G_{m,n}^M$ is complete so the diameter of the graph is 1.
Case III. Let $n < m \leq n(n-1)$ and $i,\ j,\ k$ are distinct vertices in V. The vertex $i = 1$ is adjacent to all other vertices $k \in V$ as $m \nmid 1 \cdot k$. Let the vertices j, s are not adjacent. Then the vertices j, s are connected via the vertex $i = 1$ as j is adjacent to 1 and 1 is adjacent to s. Thus, the diameter of $G_{m,n}^M$ is 2.

5. Eulerian Property of $G_{m,n}^M$

Theorem. 5.1. Let $n < m \leq n(n-1)$. The graph $G_{m,n}^M$ is not Eulerian if n is even.

Proof. Let n be even and $n < m \leq n(n-1)$. Then the degree of the vertex $i = 1$ is $n - 1$ by Theorem 2.2, which is odd, thus $G_{m,n}^M$ is not Eulerian.

According to Theorem 5.1, $G_{m,n}^M$ is not Eulerian for even integer n, so in the next results in this section we consider n as an odd integer to check whether the graph $G_{m,n}^M$ is Eulerian or not.

Lemma 5.2. Let n be odd and $m = 2 \cdot j, j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\} \subseteq V$, then $G_{m,n}^M$ is not Eulerian.

Proof. Let n be odd and $m = 2 \cdot j$, where $j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\}$. The multiples of j in $\{1, 2, \dots, n\}$ is j itself. So, the vertex $i = 2$ is not adjacent to the vertex j and itself, which implies the degree of the vertex $i = 2$ is $n - 2$, which is an odd integer. Thus $G_{m,n}^M$ is not Eulerian.

Lemma. 5.3. Let n be odd and $m = i \cdot j,\ i,j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\} \subseteq V$, then $G_{m,n}^{M}$ is not Eulerian.

Proof. Follows from Lemma 5.2, as the degree of the vertex j is $n-2$, which is odd.

Lemma 5.4. Let n be odd and $m = i \cdot j$, where $i \in \{1, 2, \dots, \lfloor\frac{n}{2}\rfloor\}$ and $j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\}$. Then $G_{m,n}^{M}$ is not Eulerian.

Proof. Let $m = i \cdot j$, where $i \in \{1, 2, \dots, \lfloor\frac{n}{2}\rfloor\}$ and $j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\}$. The vertex i is not adjacent to the vertex j or any multiples of j in $\{1, 2, \dots, n\}$. Thus, the degree of the vertex i is $deg(i) = n - \lfloor\frac{n}{j}\rfloor - 1 = n - 2$, as the number of multiples of j up to n is $\lfloor\frac{n}{j}\rfloor = 1$. Since n is odd, so $n-2$ is odd, hence $G_{m,n}^{M}$is not Eulerian.

Lemma 5.5. Let n be odd and $m = i \cdot j$, where $i,j \in \{1, 2, \dots, \lfloor\frac{n}{2}\rfloor\}$. Then $G_{m,n}^{M}$ is not Eulerian if either $\lfloor\frac{n}{i}\rfloor$ or $\lfloor\frac{n}{j}\rfloor$ is an odd integer.

Proof. Let $m = i \cdot j$, where $i,j \in \{1, 2, \dots, \lfloor\frac{n}{2}\rfloor\}$. Then in $G_{m,n}^{M}$, the vertex i is not adjacent to the multiples of j in $\{1, 2, \dots, n\}$ as well as the vertex j is not adjacent to the multiples of i in $\{1, 2, \dots, n\}$. The number of multiples of i,j up to n is $\lfloor\frac{n}{i}\rfloor$, $\lfloor\frac{n}{j}\rfloor$ respectively. Again, the vertex i is not adjacent to itself. Thus, the number of vertices not adjacent to i is $\lfloor\frac{n}{j}\rfloor + 1$ and similarly the number of vertices not adjacent to j is $\lfloor\frac{n}{i}\rfloor + 1$. But n is an odd integer. Thus $n - (\lfloor\frac{n}{j}\rfloor + 1)$ is an odd integer if $\lfloor\frac{n}{j}\rfloor$ is an odd integer. Similarly, $n - (\lfloor\frac{n}{i}\rfloor + 1)$ is an odd integer if $\lfloor\frac{n}{i}\rfloor$ is an odd integer. Hence the result follows.

Theorem 5.6. Let n be an odd integer. Then $G_{m,n}^{M}$ is not Eulerian if
(1) $m = 2 \cdot j$, where $j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\}$
(2) $m = i \cdot j$, where $i,j \in \{\lfloor\frac{n}{2}\rfloor + 1, \lfloor\frac{n}{2}\rfloor + 2, \dots, n\}$.
(3) $m = i \cdot j$, where $i \in \{1,2, \dots, \lfloor\frac{n}{2}\rfloor\}$ and $j \in \{\lfloor\frac{n}{2}\rfloor + 1, \dots, n\}$.
(4) $m = i \cdot j$, where $i,j \in \{1,2, \dots, \lfloor\frac{n}{2}\rfloor\}$ and either $\lfloor\frac{n}{i}\rfloor$ or $\lfloor\frac{n}{j}\rfloor$ is an odd integer.

Proof. Follows from Lemma 5.2, Lemma 5.3, Lemma 5.4 and Lemma 5.5.

Theorem 5.7. Let n be an odd integer and $G_{m,n}^{M}$ be a complete graph. Then $G_{m,n}^{M}$ is Eulerian.

Proof. As the graph $G_{m,n}^{M}$ is complete, so the degrees of the vertices are $n-1$, which is even as n is odd. Hence $G_{m,n}^{M}$ is Eulerian.

Theorem 5.8. Let n be an odd integer, $m = p^2$ $(m > n)$, where $p < \lfloor \frac{n}{2} \rfloor$ is a prime and $\lfloor \frac{n}{p} \rfloor$ is an odd integer, then $G_{m,n}^M$ is Eulerian.

Proof. The proof follows from Lemma 4.4.

6. **Balanced Conditions for $G_{m,n}^M$**

The average degree $d(\Gamma)$ of a graph $\Gamma = (V_\Gamma, E_\Gamma)$ is defined as $d(\Gamma) = \frac{\sum_{i=1}^{l} \deg(v_i)}{l}$, where $v_i \in V_\Gamma$ for $i = 1, 2, \dots, l$ and $l = |V_\Gamma|$ is the order of the graph Γ. In general, $d(\Gamma)$ is not necessarily an integer. The authors in [2] defined the top of a graph Γ as $\mu(\Gamma) = \lceil d(\Gamma) \rceil$, the balanced vertex set $B_\Gamma = \{v \in V_\Gamma : deg(v) = \mu(\Gamma)\}$, the upper vertex set $U_\Gamma = \{v \in V_\Gamma : deg(v) > \mu(\Gamma)\}$ and the lower vertex set as $L_\Gamma = \{v \in V_\Gamma : deg(v) < \mu(\Gamma)\}$. Γ is said to be balanced graph if $U_\Gamma = \phi$ If not, Γ is a non-balanced graph. The gap of Γ is $h(\Gamma) = l(\mu(\Gamma) - d(\Gamma))$.

Theorem. 6.1. For $t (\in \mathbb{N})$ distinct pair of vertices $i, j \in V$ such that $m = i \cdot j, (m > n)$, where $i,\ j \in \{\lfloor \frac{n}{2} + 1 \rfloor, \dots, n\}$, the top of the graph $G_{m,n}^M$ is $n - 1$. Moreover $G_{m,n}^M$ is balanced.

Proof. Let m= $i_1 \cdot j_1 = i_2 \cdot j_2 = \dots = i_t \cdot j_t$, where $i_1, j_1, i_2, j_2, \dots, i_t, j_t \in \{\lfloor \frac{n}{2} + 1 \rfloor, \dots,$ n}. Then
using Lemma 4.6, we find, $G_{m,n}^M$ contain $n - t$ vertices of degree $n - 1$ and t vertices of degree $n - 2$. So, the average degree of the graph $G_{m,n}^M$ is $d\,(G_{m,n}^M) = \frac{\{(n-t)\cdot(n-1) + t\cdot(n-2)\}}{n} = \frac{n^2 - n - t}{n} = n - 1 - \frac{t}{n}$. Hence the top of $G_{m,n}^M$ is $\mu\,(G_{m,n}^M) = \lceil d\,(G_{m,n}^M) \rceil = n - 1$, as $\lceil \frac{t}{n} \rceil = 0$. Thus, the top of $G_{m,n}^M$ is $\mu(G_{m,n}^M) = n - 1$, implies that $G_{m,n}^M$ is balanced [2].

Theorem 6.2. Let $m > n$. The graph $G_{\{m,n\}}^M$ is balanced if
(i) m is an odd prime;
(ii) m is a multiple of an odd prime $p > n$;
(iii) $\frac{m}{i} > n$, where $i \leq n$ is a factor of m and $i \in V$;
(iv) $m = p^2$, where p is an odd prime and $\lfloor n/2 \rfloor < p \leq n$.

Proof. Easily the proof follows, as $G_{m,n}^M$ is complete for all the cases by Theorem 4.1, implying $d(G_{m,n}^M) = \mu\,(G_{m,n}^M) = n - 1$.

M. P. Damas et al. [2] mentioned that there are balanced and non-regular graphs for which lower vertex set $L \neq \phi$. We observe that there are class of

$G_{m,n}^{M}$ graphs which are non-regular but balanced and the lower vertex set $L \neq \phi$.

Theorem. 6.3. For $m > n$ and $m = p^2$, where p is an odd prime and $p < \lfloor \frac{n}{2} \rfloor$, the graph $G_{m,n}^{M}$ is balanced. Moreover, the independent vertex set of $G_{m,n}^{M}$ form the lower vertex set of the given graph.

Proof. Let $m = p^2$, where p is an odd prime and $p < \lfloor \frac{n}{2} \rfloor$. Then $G_{m,n}^{M}$ contains $n - \lfloor \frac{n}{p} \rfloor$ vertices of degree $n-1$ and $\lfloor \frac{n}{p} \rfloor$ vertices of degree $n - \lfloor \frac{n}{p} \rfloor$. Thus, the average degree $d(G_{m,n}^{M})= \frac{(n-1)(n-\lfloor \frac{n}{p} \rfloor)+\lfloor \frac{n}{p} \rfloor(n-\lfloor \frac{n}{p} \rfloor)}{n} = n - 1 - \frac{\lfloor \frac{n}{p} \rfloor}{n} - \frac{\lfloor \frac{n}{p} \rfloor \lfloor \frac{n}{p} \rfloor}{n}$. Hence the top of the graph $G_{m,n}^{M}$ is $\mu\,(G_{m,n}^{M}) = \lceil d(G_{m,n}^{M}) \rceil = n - 1$, which implies $G_{m,n}^{M}$ is balanced [2]. Assume $I = \{x \in V \mid x \text{ is a multiple of } p\} \subseteq V$. Then the set I forms an independent set as for any $s_1,\ s_2 \in I,\ m \mid s_1 \cdot s_2$. The cardinality of the set I is $\lfloor \frac{n}{p} \rfloor$ and the degree of any vertex $s \in I$ is $n - \lfloor n/p \rfloor < n - 1$. Thus, the lower vertex set of $G_{m,n}^{M}$ is I.

Theorem. 6.4. For $m > n,\ m = pq$, where $p \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\},\ q \in \{\lfloor \frac{n}{2} \rfloor + 1, \dots, n\}$ are odd primes, the graph $G_{m,n}^{M}$ is balanced.

Proof. Let $m = pq$, where $p \in \left\{1,2, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\},\ q \in \{\lfloor \frac{n}{2} \rfloor + 1, \dots, n\}$ are odd primes, then the possible degrees are $\{n-1, n - \lfloor \frac{n}{q} \rfloor - 1, n - \lfloor \frac{n}{p} \rfloor - 1\}$. Thus the average degree $d(G_{m,n}^{M})= \frac{1}{n}[(n-1)(n - \lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{q} \rfloor + \lfloor \frac{n}{pq} \rfloor) + (n - \lfloor \frac{n}{q} \rfloor - 1)\lfloor \frac{n}{p} \rfloor + (n - \lfloor \frac{n}{p} \rfloor - 1)\lfloor \frac{n}{q} \rfloor] = n - 1 - \frac{2}{n}\lfloor \frac{n}{p} \rfloor \lfloor \frac{n}{q} \rfloor$. But $\frac{2}{n}\lfloor \frac{n}{p} \rfloor \lfloor \frac{n}{q} \rfloor = 0$, as $\lfloor \frac{n}{q} \rfloor = 1$ and $\lfloor \frac{n}{p} \rfloor < \lfloor \frac{n}{2} \rfloor$. Hence the Top of $G_{m,n}^{M}$ is $\mu\,(G_{m,n}^{M}) = \left\lceil d(G_{m,n}^{M}) \right\rceil = n - 1$, which follows $G_{m,n}^{M}$ is balanced.

Theorem 6.5. Let $n < m \leq n(n-1)$ and $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} + 1 \rfloor, \dots, n\} \subseteq V$ is a prime. Then $G_{m,n}^{M}$ is non-balanced, if n is even and $G_{m,n}^{M}$ is balanced, if n is odd.

Proof. Let $m = 2 \cdot p$, where p is a prime and $p \in \{\lfloor \frac{n}{2} + 1 \rfloor, \dots, n\} \subseteq V$. Using Lemma 4.3, we find the average degree $d(G_{m,n}^{M})$ and the top $\mu(G_{m,n}^{M})$ of $G_{m,n}^{M}$.
Let n be even. Then the average degree $d(G_{m,n}^{M}) = \frac{1}{n}\{\frac{n-2}{2}(n-1) + \frac{n}{2}(n-2) + \frac{n-2}{2}\} = n - 2$. The top of $G_{m,n}^{M}$ is $\mu\left(G_{m,n}^{M}\right) = \lceil d(G) \rceil = \lceil n - 2 \rceil = n - 2$. But the vertex $w = 1 \in V$ is of degree $n - 1$, which implies $G_{m,n}^{M}$ is a non-balanced graph [2].

Let n be odd. Then the average degree $d(G_{m,n}^{M}) = 1/n\{\frac{n-1}{2}(n-1) + \frac{n-1}{2}(n-2) + \frac{n-1}{2}\} = n - 2 + \frac{1}{n}$. Thus, the top of $G_{m,n}^{M}$ is $\mu(G_{m,n}^{M}) = \lceil d(G) \rceil = \left\lceil n - 2 + \frac{1}{n} \right\rceil = n - 2 + \left\lceil \frac{1}{n} \right\rceil = n - 1$. Hence the result follows.

As a consequence of Theorem 6.5, we find that the gap of the non-regular graph $G_{m,n}^{M}$ is zero.

Corollary 6.6. Let $n < m \leq n(n-1)$ and $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2}+1 \rfloor, \dots, n\} \subseteq V$ is a prime. Then the gap of $G_{m,n}^{M}$ is zero if n is even.

Proof. Let n be even and and $m = 2 \cdot p$, where $p \in \{\lfloor \frac{n}{2} + 1 \rfloor, \dots, n\} \subseteq V$ is a prime. Then from Theorem 6.5, for the graph $G_{m,n}^{M}$, $\mu(G_{m,n}^{M}) = \lceil d(G) \rceil = \lceil n - 2 \rceil = n - 2$. Thus, the gap $h(G_{m,n}^{M}) = n(\mu(G_{m,n}^{M}) - d(G_{m,n}^{M})) = 0$.

7. Conclusion

In this paper, we have defined and studied an undirected graph $G_{m,n}^{M} = (V, E)$, where the vertex set $V = \{1, 2, \dots, n\}$ for $n, m \in \mathbb{N}$ and two distinct vertices $i, j \in V$ are adjacent if and only if $m \nmid i \cdot j$. We observed that $G_{m,n}^{M}$ is disconnected if $m \leq n$ and $G_{m,n}^{M}$ is complete for $m > n(n-1)$. We studied vertex degree, edge degree, diameter, Eulerian property of $G_{m,n}^{M}$ for various values of m. We found that $G_{m,n}^{M}$ is a class of split graph for $m = p^2 > n$, where $p \leq \lfloor \frac{n}{2} \rfloor$ and for $m = i \cdot j$, where i, j are the unique factors of m such that $i, j \in \{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \dots, n\} \subseteq V$. We also observed that there are non-regular $G_{m,n}^{M}$ graphs which are balanced.

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