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FUZZY GENERALIZED β-CONTINUITY AND FUZZY GENERALIZED β-γ-CONTINUITY IN FUZZY TOPOLOGICAL SPACES

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Abstract. This study is aimed to introduce a new type of fuzzy generalized continuities called fuzzy generalized β-continuity and fuzzy generalized β-γ-continuity. We also formulate the definitions of fuzzy generalized β-open maps, fuzzy generalized β-γ-open maps, fuzzy generalized β-irresolute and fuzzy generalized β-γ-irresolute in fuzzy topological spaces. We prove that every fuzzy continuous map is fgβ-continuous as well as that fgβ-irresolute map is fgβ-continuous. Also we show that every fuzzy continuous map is $fg\beta_\gamma$-continuous as well as that every $fg\beta_\gamma$-irresolute map is $fg\beta_\gamma$-continuous. Furthermore, we demonstrate some of their properties and theorems.

Keywords: Fuzzy β-γ-open, fuzzy generalized β-γ-closed set, fuzzy generalized β-continuous, fuzzy generalized β-γ-continuous, fuzzy generalized β-γ-open map.

AMS Subject Classification: 54A40, 54C10, 54C20.

1. Introduction

In 1965, Zadeh [8] introduced the notion of fuzzy sets and fuzzy sets operations. The fuzzy concept has invaded almost all branches of mathematics. Fuzzy sets have applications in many fields such as decision making, pattern recognition, medical diagnosis problems. The notion of fuzzy topology was introduced by Chang [2]. Since then, many notions in the general topology such as generalized closed set, generalized open set, generalized continuous, generalized irresolute, generalized open map, generalized closed maps have extended to fuzzy topological spaces. Kasahara [4] defined an operation γ in topological spaces. The notion of γ-open fuzzy set was introduced by Kalitha and Das[3]. Balasubramanian [1] initiated the notion of β-open fuzzy sets. Fuzzy β-γ-open set was introduced by Sivashanmugaraja and Vadivel [6]. A mapping is continuous specifically if its pre images of open sets are again open sets. In this paper, we introduce and investigate the properties of fuzzy generalized β-continuity and fuzzy generalized β-γ-continuity in fuzzy topological spaces. Notations, definitions and preliminaries are given the section 2. The main results of the paper appear in section 3 and 4. In section 3, we prove that every fuzzy continuous map is fgβ-continuous as well as that fgβ-irresolute mapping is fgβ-continuous. Also we show that the composition of two fuzzy generalized β-γ-continuous mappings need not be fuzzy generalized β-γ-continuous. Finally we investigate the relationship between fuzzy generalized β-continuity and fuzzy generalized β-γ-continuity maps.

2. Preliminaries

Throughout this paper, (X, τ) or X always mean a fuzzy topological space. By $\underline{0}$ and $\underline{1}$, we mean the constant fuzzy sets taking on the values 0 and 1 on X respectively. Now we recall the following basic definitions which we used in this paper.

Definition 2.1. [1] Let λ be any fuzzy set in a fts X. Then λ is said to be fuzzy β-open if $\lambda \leq cl(int(cl(\lambda)))$. The complement of a fuzzy β-open set is said to be fuzzy β-closed.

Definition 2.2. [1] Let λ be a fuzzy set in a fts X. Then the β-interior of λ is defined as $\beta int(\lambda) = \vee\{\mu : \mu \leq \lambda, \mu \in F\beta O(X)\}$ and the β-closure of λ is defined as $\beta cl(\lambda) = \wedge\{\mu : \mu \geq \lambda, \mu \in F\beta C(X)\}$.

Definition 2.3. [6] Let λ be any fuzzy set in a fts X. Then λ is said to be fuzzy β-γ-open if $\lambda \leq cl(\tau_\gamma\text{-}int(cl(\lambda)))$. The complement of a fuzzy β-γ-open set is said to be fuzzy β-γ-closed.

Definition 2.4. [5] Let λ be a fuzzy set in a fts X. Then the β-γ-interior of λ is defined as $\beta int_\gamma(\lambda) = \vee\{\mu: \mu \leq \lambda, \mu \in F\beta_\gamma O(X)\}$ and the β-γ-closure of λ is defined as $\beta cl_\gamma(\lambda) = \wedge\{\mu: \mu \geq \lambda, \mu \in F\beta_\gamma C(X)\}$.

Definition 2.5. [7] A fuzzy subset λ of a fts X is said to be fuzzy generalized β-closed if $\beta cl(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is fuzzy open in X. The complement of fuzzy generalized β-closed set is said to be fuzzy generalized β-open.

Definition 2.6. [2] A mapping $\theta: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is called
(1) fuzzy continuous, if $\theta^{-1}(\mu)$ is a fuzzy open set of X, for every fuzzy open set of Y;
(2) fuzzy open, if for every fuzzy open set μ in X, θ(μ) is fuzzy open in Y;
(3) fuzzy closed, if for every fuzzy closed set μ in X, θ(μ) is fuzzy closed in Y.

3. Fuzzy Generalized β-continuous Mapping

In this section, we introduce fuzzy generalized β-continuous maps, fuzzy generalized β-irresolute maps, fuzzy generalized β-closed maps and fuzzy generalized β-open maps in fuzzy topological spaces and study some of their properties.

Definition 3.1. A mapping $\theta: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is called fuzzy generalized β-continuous (briefly, fgβ-continuous), if the inverse image of every closed fuzzy set of Y is gβ-closed fuzzy set of X.

Theorem 3.1. A mapping $\theta : (X, \tau_X) \to (Y, \tau_Y)$ is fgβ-continuous iff $\theta^{-1}(\lambda)$ is gβ-open fuzzy set in X, for every open fuzzy set λ in Y.

Proof. Let λ be an open fuzzy set of Y. Then 1-λ is a gβ-closed fuzzy set of Y. Since θ is fgβ-continuous, we obtain $\theta^{-1}(1-\lambda) = 1-\theta^{-1}(\lambda)$ is gβ-closed fuzzy set of X. Thus $\theta^{-1}(\lambda)$ is gβ-open fuzzy set of X.
Converse is obvious.

Theorem 3.2. If $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is fgβ-continuous, then
(i) For every fuzzy point x_α of X and every λ in Y such that $\theta(x_\alpha)\ q\ \lambda$, there exists a gβ-open fuzzy set μ of X such that $x_\alpha \in \mu$ and $\theta(\mu) \leq \lambda$.
(ii) For every fuzzy point x_α of X and every λ in Y such that $\theta(x_\alpha)\ q\ \lambda$, there exists a gβ-open fuzzy set μ of X such that $x_\alpha\ q\ \mu$ and $\theta(\mu) \leq \lambda$.

Proof. (i) Let x_α be a fuzzy point of X, then $\theta(x_\alpha)$ is a fuzzy point of Y. Now let λ be an open fuzzy set in Y such that $\theta(x_\alpha)\ q\ \lambda$. Take $\mu = \theta^{-1}(\lambda)$. Since θ is fgβ-continuous, we obtain μ is gβ-open fuzzy set of X and $x_\alpha \in \mu$, $\theta(\mu) = \theta(\theta^{-1}(\lambda)) \leq \lambda$.
(ii) Let x_α be a fuzzy point of X and let λ in Y such that $\theta(x_\alpha)\ q\ \lambda$. Take $\mu = \theta^{-1}(\lambda)$. Since θ is fgβ-continuous, we obtain μ is gβ-open fuzzy set of X, such that $x_\alpha\ q\ \mu$ and $\theta(\mu) = \theta(\theta^{-1}(\lambda)) \leq \lambda$.

Theorem 3.3. Every fuzzy continuous map is fgβ-continuous

Proof. Let $\theta: (X, \tau_X) \to (Y, \tau_Y)$ be a fuzzy continuous map. Let λ be an open fuzzy set in Y. Since θ is fuzzy continuous, $\theta^{-1}(\lambda)$ is an open fuzzy set in X. And therefore $\theta^{-1}(\lambda)$ is gβ-open fuzzy set in X. Thus θ is fgβ-continuous map.

The converse of the above theorem 3.3 need not be true as shown in the following example.

Example 3.1. Let X=Y={a, b, c} and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ defined by $\lambda_1 = \underline{0.3}$, $\lambda_2 = \underline{0.4}$, $\lambda_3(a) = 0.8$, $\lambda_3(b) = 0.7$, $\lambda_3(c) = 0.5$, $\lambda_4(a) = 0.5$, $\lambda_4(b) = 0.6$, $\lambda_4(c) = 0.3$. Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$ and $\tau_Y = \{\underline{1}, \underline{0}, \lambda_3, \lambda_4\}$. Then (X, τ_X) and (Y, τ_Y) are fts. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ defined as θ(a)=b, θ(b)=c, θ(c)=a. Then θ is fgβ-continuous map but not fuzzy continuous.

Definition 3.2. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy β-generalized irresolute (briefly fgβ-irresolute), if the inverse image of every gβ-closed fuzzy set of Y is gβ-closed fuzzy set of X.

Theorem 3.4. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is fgβ-irresolute iff $\theta^{-1}(\lambda)$ is gβ-open fuzzy set in X, for every gβ-open fuzzy set λ in Y.

Theorem 3.5. Every fgβ-irresolute mapping is fgβ-continuous.

Proof. Let θ: $(X, \tau_X) \to (Y, \tau_Y)$ is fgβ-irresolute . Let μ be a closed fuzzy set in Y. Then clearly μ is gβ-closed fuzzy set in Y. Since θ is fgb- irresolute, we have $\theta^{-1}(\mu)$ is a gβ-closed fuzzy set in X. Hence θ is fgβ-continuous.

The converse of the above theorem 3.5 need not be true as shown in the following example.

Example 3.2. Let X=Y={a, b, c} and $\lambda_1, \lambda_2 \in I^X$ defined by $\lambda_1 = \underline{0.3}$, $\lambda_2 = \underline{0.4}$. Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$ and $\tau_Y = \{\underline{1}, \underline{0}, \lambda_2\}$. Then (X, τ_X) and (Y, τ_Y) are fts. Let θ: $(X,\tau_X) \to (Y,\tau_Y)$ be an identity mapping. Then θ is fgβ-continuous map but not fgβ-irresolute.

Theorem 3.6. Let θ_1: $(X, \tau_X) \to (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings.
(i) If θ_1 is fgβ-continuous and θ_2 is fuzzy continuous, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ-continuous.
(ii) If θ_1 and θ_2 are fgβ-irresolute mappings, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ-irresolute.
(iii) If θ_1 is fgβ-irresolute and θ_2 is fgβ-continuous, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ-continuous.

Proof. (i) Let μ be a closed fuzzy subset of Z. Since θ_2 is fuzzy continuous, we obtain $\theta_2^{-1}(\mu)$ is closed fuzzy set of Y. Now θ_1 is fgβ-continuous and $\theta_2^{-1}(\mu)$ is closed fuzzy set of Y, so by definition of fgβ-continuous, $\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ-closed fuzzy set of X. Thus $(\theta_2 \circ \theta_1)$ is fgβ-continuous.
(ii) Let μ be gβ-closed fuzzy subset of Z. Since θ_2 is fgβ-irresolute, we obtain $\theta_2^{-1}(\mu)$ is gβ-closed fuzzy set of Y. Also θ_1 is fgβ-irresolute, so $\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ-closed fuzzy set of X. Hence $(\theta_2 \circ \theta_1)$ is fgβ-irresolute.
(iii) Let μ be a closed fuzzy subset of Z. Since θ_2 is fgβ-continuous, $\theta_2^{-1}(\mu)$ is gβ-closed fuzzy subset of Y. Now $\theta_1$is fgβ-irresolute, so inverse image of each gβ-closed fuzzy set of Y is gβ-closed fuzzy set of X. Thus $\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ-closed fuzzy set of X. Thus $(\theta_2 \circ \theta_1)$ is fgβ-continuous.

Definition 3.3. A mapping θ: $(X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy generalized β-open (briefly fgβ-open), if the image of each open fuzzy set in X is gβ-open fuzzy set in Y.

Definiton 3.4. A mapping θ: $(X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy generalized gβ-closed (briefly fgβ-closed), if the image of each closed fuzzy set in X is gβ-closed fuzzy set in Y.

Example 3.3. Let X=Y={a, b, c} and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ defined by $\lambda_1 = \underline{0.2}$, $\lambda_2 = \underline{0.5}$. $\lambda_3(a) = 0.8$, $\lambda_3(b) = 0.7$, $\lambda_3(c) = 0.5$, $\lambda_4(a) = 0.5$, $\lambda_4(b) = 0.6$, $\lambda_4(c) = 0.3$. Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$ and $\tau_Y = \{\underline{1}, \underline{0}, \lambda_3, \lambda_4\}$. Then (X, τ_X) and (Y, τ_Y) are fts. Let θ: $(X, \tau_X) \to (Y, \tau_Y)$ be an identity mapping. Then θ is fgβ-open and fgβ-closed map.

Theorem 3.7. If θ_1: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ is fuzzy closed and $\theta_2 : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$ is fgβ-closed , then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \rightarrow (Z, \tau_Z)$ is fgβ-closed .

Proof. Let λ be a closed fuzzy set in X. Since θ_1 is fuzzy closed, we obtain $\theta_1(\lambda)$ is closed fuzzy set in Y. Also θ_2 is fgβ-closed, $\theta_2(\theta_1(\lambda))$ is gβ-closed fuzzy set in Z. But $\theta_2(\theta_1(\lambda)) = (\theta_2 \circ \theta_1)(\lambda)$ is gβ-closed fuzzy set in Z. Therefore $(\theta_2 \circ \theta_1)$ is fgβ-closed.

Theorem 3.8. A mapping $\theta : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is fgβ-closed iff for every fuzzy set λ of Y and for every open fuzzy set μ such that $\theta^{-1}(\lambda) \leq \mu$, there is a gβ-open fuzzy set η of Y such that $\lambda \leq \eta$ and $\theta^{-1}(\eta) \leq \mu$.

Proof. Let θ be a fgβ-closed map. Let λ be a fuzzy set of Y and μ be an open fuzzy set of X such that $\theta^{-1}(\lambda) \leq \mu$. Then $\eta = 1\text{-}f(1\text{-}\mu)$ is a gβ-open fuzzy set in Y such that $\lambda \leq \eta$ and $\theta^{-1}(\eta) \leq \mu$.

Conversely, suppose that μ is a closed fuzzy set of X. Then $\theta^{-1}(1\text{-}\theta(\mu)) \leq 1\text{-}\mu$. and $1\text{-}\mu$ is an open fuzzy set. By hypothesis, there is a gβ-open fuzzy set η of Y such that $1\text{-}\theta(1\text{-}\eta) \leq \lambda$ and $\theta^{-1}(\lambda) \leq 1\text{-}\mu$. So $\mu \leq 1\text{-} \theta^{-1}(\lambda)$. Thus $1\text{-}\lambda \leq \theta(\lambda) \leq \theta(1\text{-}\theta^{-1}(\lambda)) \leq 1\text{-}\lambda$. which implies $\theta(\mu) = 1\text{-}\lambda$. Since $1\text{-}\lambda$ is gβ-closed fuzzy set, $\theta(\mu)$ is gβ-closed fuzzy set and so θ is a fgβ-closed map.

Theorem 3.9. Let θ_1: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \rightarrow (Z, \tau_Z)$ be two mappings such that $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \rightarrow (Z, \tau_Z)$ is fgβ-closed map

(i) If θ_1 is fuzzy continuous and surjective, then θ_2 is fgβ-closed map.

(ii) If θ_2 is fgβ-irresolute and injective, then θ_1 is fgβ-closed map.

Proof. (i) Let μ be a closed fuzzy set of Y. Since θ_1 is fuzzy continuous, we obtain $\theta^{-1}(\mu)$ is closed fuzzy set in X. Also $(\theta_2 \circ \theta_1)$ is fgβ-closed map, then $(\theta_2 \circ \theta_1)(\theta^{-1}(\mu)) = \theta_2(\mu)$ is gβ-closed fuzzy set in Z. Thus θ_2 is fgβ-closed map.

(ii) Let μ be a closed fuzzy set in X. Then $(\theta_2 \circ \theta_1)(\mu)$ is gβ-closed fuzzy set in Z and so $\theta_2^{-1}(\theta_2 \circ \theta_1)(\mu)) = \theta(\mu)$ is gβ-closed fuzzy set in Y. Since θ_2 is fgβ-irresolute and injective. Thus θ is a fgβ-closed map.

Theorem 3.10. If λ is gβ-closed fuzzy set in X and θ: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ is bijective, fuzzy continuous and fgβ-closed , then $\theta(\lambda)$ is gβ-closed fuzzy set in Y.

Proof. Let $\theta(\lambda) \leq \mu$, where μ be an open fuzzy set in Y. Since θ is fuzzy continuous, $\theta^{-1}(\mu)$ is an open fuzzy set in X containing λ. Thus $\beta cl(\lambda) \leq \theta^{-1}(\mu)$ as λ is gβ-closed fuzzy set. Since θ is fgβ-closed, $\theta(\beta cl(\lambda))$ is gβ-closed fuzzy set contained in the open fuzzy set, which implies $\beta cl(\theta(\beta cl(\lambda))) \leq \mu$ and thus $\beta cl(\theta(\lambda)) \leq \mu$. So $\theta(\lambda)$ is gβ-closed fuzzy set in Y.

4. Fuzzy Generalized β-γ-continuous mapping

In this section, we introduce fuzzy generalized β-γ-closed sets, fuzzy generalized β-γ-continuous maps and fuzzy generalized β-γ-irresolute maps in fuzzy topological spaces. Also we discuss the relation between fuzzy generalized β-continuity and fuzzy generalized β-γ-continuity.

Definition 4.1. A fuzzy subset λ of a fts X is said to be fuzzy generalized β-γ-closed (briefly $fg\beta_\gamma$-closed) if $\beta cl_\gamma(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is fuzzy open in X. The complement of fuzzy generalized β-γ-closed set is said to be fuzzy generalized β-γ-open.

Definition 4.2. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy generalized β-γ-continuous (briefly $fg\beta_\gamma$-continuous), if the inverse image of every closed fuzzy set of Y is $g\beta_\gamma$-closed fuzzy set of X.

Theorem 4.1. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is $fg\beta_\gamma$-continuous iff $\theta^{-1}(\lambda)$ is $g\beta_\gamma$-open fuzzy set in X, for every open fuzzy set λ in Y.

Proof. Let λ be an open fuzzy set of Y. Then 1-λ is a $g\beta_\gamma$-closed fuzzy set of Y. Since θ is $fg\beta_\gamma$-continuous, we obtain $\theta^{-1}(1\text{-}\lambda) = 1\text{-}\theta^{-1}(\lambda)$ is $g\beta_\gamma$-closed fuzzy set of X. Thus $\theta^{-1}(\lambda)$ is $g\beta_\gamma$-open fuzzy set of X. Converse is obvious.

Theorem 4.2. Every fuzzy continuous map is $fg\beta_\gamma$-continuous.

Proof. Let $\theta : (X, \tau_X) \to (Y, \tau_Y)$ be a fuzzy continuous map. Let λ be an open fuzzy set in Y. Since θ is fuzzy continuous, $\theta^{-1}(\lambda)$ is an open fuzzy set in X. And therefore $\theta^{-1}(\lambda)$ is $g\beta_\gamma$-open fuzzy set in X. Thus θ is $fg\beta_\gamma$-continuous map.

The converse of the above theorem 4.2 need not be true as shown in the following example.

Example 4.1. Let X=Y={a, b, c} and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ defined by $\lambda_1 = \underline{0.7}$, $\lambda_2 = \underline{0.8}$, $\lambda_3(a) = 0.5$, $\lambda_3(b) = 0.4$, $\lambda_3(c) = 0.9$, $\lambda_4(a) = 0.6$, $\lambda_4(b) = 0.5$, $\lambda_4(c) = 0.9$. Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$ and $\tau_Y = \{\underline{1}, \underline{0}, \lambda_3, \lambda_4\}$. Then (X, τ_X) and (Y, τ_Y) are fts. Define an operation γ on τ_X by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_1) = \lambda_1$, $\gamma(\lambda_2) = \underline{0.9}$ A mapping $\theta: (X,\tau_X) \to (Y, \tau_Y)$ be defined as θ(a)=b, θ(b)=a, θ(c)=c. Then θ is $fg\beta_\gamma$-continuous map but not fuzzy continuous.

Definition 4.3. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y)$ is called fuzzy generalized β-γ-irresolute (briefly $fg\beta_\gamma$-irresolute), if the inverse image of every $g\beta_\gamma$-closed fuzzy set of Y is $g\beta_\gamma$-closed fuzzy set of X.

Theorem 4.3. A mapping $\theta: (X, \tau_X) \to (Y, \tau_Y))$ is $fg\beta_\gamma$-irresolute iff $\theta^{-1}(\lambda)$ is $g\beta_\gamma$-open fuzzy set in X, for every $g\beta_\gamma$-open fuzzy set λ in Y.

Theorem 4.4. Every $fg\beta_\gamma$-irresolute mapping is $fg\beta_\gamma$-continuous.

Proof. Let θ: $(X, \tau_X) \to (Y, \tau_Y)$ is fgβ_γ-irresolute. Let μ be a closed fuzzy set in Y. Then clearly μ is gβ_γ-closed fuzzy set in Y. Since θ is fgβ_γ-irresolute, we have $\theta^{-1}(\mu)$ is a gβ_γ-closed set in X. Hence θ is fgβ_γ-continuous.

The converse of the above theorem 4.4 need not be true as shown in the following example.

Example 4.2. Consider an Example 4. Define an operation γ on τ_X by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_1) = \lambda_1$, $\gamma(\lambda_2) = \underline{cl}(\lambda_2)$ and define an operation γ on τ_Y by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_1) = \lambda_1$, $\gamma(\lambda_2) = \underline{cl}(\lambda_2)$. Let θ: $(X, \tau_X) \to (Y, \tau_Y)$ be an identity mapping. Then θ is fgβ_γ-continuous map but not fgβ_γ-irresolute.

Theorem 4.5. Let θ_1: $(X, \tau_X) \to (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings.
(i) If θ_1 is fg$\beta_\gamma$continuous and θ_2 is fuzzy continuous, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ_γ-continuous.
(ii) If θ_1 and θ_2 are fgβ_γ-irresolute mappings, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ_γ -irresolute.
(iii) If θ_1 is fgβ_γ-irresolute and θ_2 is fgβ_γ-continuous, then $(\theta_2 \circ \theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ_γ-continuous.

Proof. (i) Let μ be a closed fuzzy subset of Z. Since θ_2 is fuzzy continuous, we obtain $\theta_2^{-1}(\mu)$ is closed fuzzy set of Y. Now θ_1 is fgβ_γ-continuous and $\theta_2^{-1}(\mu)$ is closed fuzzy set of Y, so by definition of fgβ_γ-continuous,
$\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ_γ-closed fuzzy set of X. Thus $(\theta_2 \circ \theta_1)$ is fgβ_γ-continuous.
(ii) Let μ be gβ_γ-closed fuzzy subset of Z. Since θ_2 is fgβ_γ-irresolute, we obtain $\theta_2^{-1}(\mu)$ is gβ_γ-closed fuzzy set of Y. Also θ_1 is fgβ_γ-irresolute, so
$\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ_γ-closed fuzzy set of X. Hence $(\theta_2 \circ \theta_1)$ is fgβ_γ-irresolute.
(iii) Let μ be a closed fuzzy subset of Z. Since θ_2 is fgβ_γ-continuous, $\theta_2^{-1}(\mu)$ is gβ_γ-closed fuzzy subset of Y. Since θ_1 is fgβ_γ-irresolute, inverse image of each gβ_γ-closed fuzzy set of Y is gβ_γ-closed fuzzy set of X.
Thus $\theta_1^{-1}(\theta_2^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is gβ_γ-closed fuzzy set of X. Thus $(\theta_2 \circ \theta_1)$ is fgβ_γ-continuous.

Definition 4.4. A mapping θ: $(X, \tau_X) \to (Y, \tau_Y)$ is called
(i) fuzzy generalized β-γ-open (briefly fgβ_γ-open), if the image of each open fuzzy set in X is gβ_γ-open fuzzy set in Y:
(ii) fuzzy generalized β-γ-closed (briefly fgβ_γ-closed), if the image of each closed fuzzy set in X is gβ_γ-closed fuzzy set in Y.

Theorem 4.6. If θ_1: $(X, \tau_X) \to (Y, \tau_Y)$ is fuzzy closed and θ_2: $(Y, \tau_Y) \to (Z, \tau_Z)$ is fgβ_γ-closed, then $(\theta_2\circ\theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ_γ-closed .

Proof. Let λ be a closed fuzzy set in X. Since θ_1 is fuzzy closed, we obtain $\theta_1(\lambda)$ is closed fuzzy set in Y. Also θ_2 is fgβ_γ-closed, so $\theta_2(\theta_1(\lambda))$ is gβ_γ-closed fuzzy set in Z. But $\theta_2(\theta_1(\lambda)) = (\theta_2\circ\theta_1)(\lambda)$. So $(\theta_2\circ\theta_1)(\lambda)$ is gβ_γ-closed fuzzy set in Z. Therefore $(\theta_2\circ\theta_1)$ is fgβ_γ-closed.

Theorem 4.7. A mapping θ: $(X, \tau_X) \to (Y, \tau_Y)$ is fgβ_γ-closed iff for every fuzzy set λ of Y and for every open fuzzy set μ such that $\theta^{-1}(\lambda) \leq \mu$, there is a g$\beta_\gamma$-open fuzzy set η of Y such that $\lambda \leq \eta$ and $\theta^{-1}(\eta) \leq \mu$.

Proof. Let θ be a fgβ_γ-closed map. Let λ be a fuzzy set of Y and μ be an open fuzzy set of X such that $\theta^{-1}(\lambda) \leq \mu$. Then $\eta = 1\text{-}\theta(1\text{-}\mu)$ is a gβ_γ-open fuzzy set in Y such that $\lambda \leq \eta$ and $\theta^{-1}(\eta) \leq \mu$.
Conversely, suppose that μ is a closed fuzzy set of X. Then $\theta^{-1}(1\text{-}\theta(\mu)) \leq \mu$ and $1\text{-}\mu$ is an open fuzzy set. By hypothesis, there is a gβ_γ-open fuzzy set η of Y such that $1\text{-}\theta(1\text{-}\eta) \leq \lambda$ and $\theta^{-1}(\lambda) \leq 1\text{-}\mu$. So $\mu \leq \theta^{-1}(\lambda)$.
Thus $1\text{-}\lambda \leq \theta(\lambda) \leq \theta(1\text{-}\theta^{-1}(\lambda)) \leq 1\text{-}\lambda$, which implies $\theta(\mu) = 1\text{-}\lambda$. Since $1\text{-}\lambda$ is gβ_γ-closed fuzzy set, $\theta(\mu)$ is gβ_γ-closed fuzzy set and so θ is a fgβ_γ-closed map.

Theorem 4.8. Let θ_1: $(X, \tau_X) \to (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings such that $(\theta_2\circ\theta_1)$: $(X, \tau_X) \to (Z, \tau_Z)$ is fgβ_γ-closed map
(i) If θ_1 is fuzzy continuous and surjective, then θ_2 is fgβ_γ-closed map.
(ii) If θ_2 is fgβ_γ-irresolute and injective, then θ_1 is fgβ_γ-closed map.

Proof. (i) Let μ be a closed fuzzy set of Y: Since θ_1 is fuzzy continuous, we obtain $\theta_1^{-1}(\mu)$ is closed fuzzy set in X. Also $(\theta_2\circ\theta_1)$ is fgβ_γ-closed map, then $(\theta_2\circ\theta_1)(\theta_1^{-1}(\mu)) = \theta_2(\mu)$ is gβ_γ-closed fuzzy set in Z. Thus θ_2 is fgβ_γ-closed map.
(ii) Let μ be a closed fuzzy set in X. Then $(\theta_2\circ\theta_1)(\mu)$ is gβ_γ-closed fuzzy set in Z and so $\theta_2^{-1}(\theta_2\circ\theta_1)^1(\mu) = (\theta_1)(\mu)$ is gβ_γ-closed fuzzy set in Y. Since θ_2 is fgβ_γ-irresolute and injective. Thus θ_1 is a fgβ_γ-closed map.

Theorem 4.9. If λ is gβ_γ-closed fuzzy set in X and θ: $(X, \tau_X) \to (Y, \tau_Y)$ is bijective, fuzzy continuous and fgβ_γ-closed, then $\theta(\lambda)$ is gβ_γ-closed fuzzy set in Y.

Proof. Let $\theta(\lambda) \leq \mu$, where μ is an open fuzzy set in Y. Since θ is fuzzy continuous, $\theta^{-1}(\mu)$ is an open fuzzy set in X containing λ. Thus $\beta cl(\lambda) \leq \theta^{-1}(\mu)$ as λ is gβ_γ-closed fuzzy set. Since θ is fgβ_γ-closed, $\theta(\beta cl(\lambda))$ is gβ_γ-closed fuzzy set contained in the open fuzzy set μ, which implies $\beta cl(\theta(\beta cl(\lambda))) \leq \mu$ and thus $\beta cl(\theta(\lambda)) \leq \mu$. So $\theta(\lambda)$ is gβ_γ-closed fuzzy set in Y.

Theorem 4.10. Every fuzzy generalized β-γ-continuous map is fuzzy generalized β-continuous.

Proof. Let θ: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ be mapping. Let λ be a closed fuzzy set in Y. Since θ is fuzzy generalized β-γ-continuous, we have $\theta^{-1}(\lambda)$ is $g\beta_\gamma$-closed fuzzy set in X. Then clearly $\theta^{-1}(\lambda)$ is gβ-closed fuzzy set in X. Thus θ is fuzzy generalized β-continuous.

The converse of the above theorem 4.10 need not be true as shown in the following example.

Example 4.3. Let X=Y={a, b, c} and $\lambda_1, \lambda_2 \in I^X$ defined by $\lambda_1 = \underline{0.4}$; $\lambda_2 = \underline{0.5}$.
Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1\}$ and $\tau_Y = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$. Then (X, τ_X) and (Y, τ_Y) are fts.
Define an operation γ on τ_X by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_1) = cl(\lambda_1)$.
Let θ: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ be an identity mapping. Then θ is fgβ-continuous map but not $fg\beta_\gamma$-continuous.

Remark 4.1. The composition of two fuzzy generalized β-continuous mappings need not be fuzzy generalized β-continuous as shown in the following example.

Example 4.4. Let X=Y=Z={a, b, c} and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$
defined by $\lambda_1 = \underline{0.7}$; $\lambda_2 = \underline{0.4}$; $\lambda_3 = \underline{0.6}$; $\lambda_4 = \underline{0.2}$.
Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1, \lambda_2\}$ $\tau_Y = \{\underline{1}, \underline{0}, \lambda_3\}$ and $\tau_Z = \{\underline{1}, \underline{0}, \lambda_4\}$.Then clearly (X, τ_X), (Y, τ_Y) and (Z, τ_Z) are fts. Let $\theta_1 : (X, \tau_X) \rightarrow (Y, \tau_Y)$ and $\theta_2 : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$ be identity mappings. Then, θ_1 and θ_2 are fuzzy generalized β-continuous mappings but $(\theta_2 \circ \theta_1)$ is not fuzzy generalized β-continuous. Since, λ_4 is an open fuzzy set of (Z, τ_Z) but $(\theta_2 \circ \theta_1)(\lambda_4)$ is not a gβ-open fuzzy set of X.

Remark 4.2. The composition of two fuzzy generalized β-γ-continuous mappings need not be fuzzy generalized β-γ-continuous as shown in the following example.

Example 4.5. Let X=Y=Z={a, b, c} and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$
defined by $\lambda_1 = \underline{0.7}$; $\lambda_2 = \underline{0.4}$; $\lambda_3 = \underline{0.6}$; $\lambda_4 = \underline{0.3}$.
Let $\tau_X = \{\underline{1}, \underline{0}, \lambda_1\}$, $\tau_Y = \{\underline{1}, \underline{0}, \lambda_2, \lambda_3\}$ and $\tau_Z = \{\underline{1}, \underline{0}, \lambda_4\}$. Then clearly (X, τ_X), (Y, τ_Y) and (Z, τ_Z) are fts.
Define an operation γ on τ_X by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_1) = cl(\lambda_1)$ and also define an operation γ on τ_Y by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\lambda_2) = \lambda_2$, $\gamma(\lambda_3) = cl(\lambda_3)$.
Let θ_1: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \rightarrow (Z, \tau_Z)$ be identity mappings. Then, θ_1 and θ_2 are fuzzy generalized β-γ-continuous mappings but $(\theta_2 \circ \theta_1)$ is not fuzzy generalized β-γ-continuous. Since, λ_4 is an open fuzzy set of (Z, τ_Z), but $(\theta_2 \circ \theta_1)(\lambda_4)$ is not a $g\beta_\gamma$-open fuzzy set of X.

Theorem 4.11. Let θ_1: $(X, \tau_X) \rightarrow (Y, \tau_Y)$ and θ_2: $(Y, \tau_Y) \rightarrow (Z, \tau_Z)$ be two mappings. If θ_1 is fuzzy generalized β-irresolute and θ_2 is fuzzy generalized β-γ-continuous mapping, then $(\theta_2 \circ \theta_1)$ is fuzzy generalized β-continuous.

Proof. Let μ be a closed fuzzy set of Z. Since θ_2 is fuzzy generalized β-γ-continuous, we obtain, $\theta_2^{-1}(\mu)$ is $g\beta_\gamma$-closed fuzzy set in Y. Then clearly $\theta_2^{-1}(\mu)$ is gβ-closed fuzzy set in Y. By hypothesis, θ_1 is fuzzy gβ-irresolute, we obtain $\theta_1^{-1}(\theta_1)^{-1}(\mu)) = (\theta_2 \circ \theta_1)^{-1}(\mu)$ is $g\beta_\gamma$-closed fuzzy set in X. Thus $(\theta_2 \circ \theta_1)$ is fgβ-continuous map.

5. Conclusion

In this paper, we introduce and investigated the notions of generalized β-continuity, generalized β-γ-continuity, generalized β-open map, generalized β-γ-open map, generalized β-irresolute and generalized β-γ-irresolute in fuzzy settings. We proved that every fuzzy generalized β-γ-continuous map is fuzzy generalized β-continuous but not converse. We showed that the composition of two fuzzy generalized β-γ-continuous (fuzzy generalized β-continuous) mappings need not be fuzzy generalized β-γ-continuous (fuzzy generalized β-continuous). There is a scope to study and extend these newly defined mappings.

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REFERENCES

1. Balasubramanian G. On Fuzzy β-compact Spaces and Fuzzy β-Exampletremely disconnected Spaces, Kybernetika., V.33, 1997, pp.271-277.
2. Chang C.L. Fuzzy topological spaces, J. Math. Anal. Appl., V.24, 1968, pp.182-189.
3. Kalita B., Das N.R. Some Aspects of Fuzzy operations, The Journal of Fuzzy Mathematics, V.19 N.3, 2011, pp.531-540.
4. Kasahara S. Operation-Compact spaces, Math. Japonica, V.24, 1979, pp.97-105.
5. Sivashanmugaraja C., Antony Jerold S. Fuzzy β-γ-open Sets and Fuzzy β-γ-open Mappings, Submitted.
6. Sivashanmugaraja C., Vadivel A. Weak forms of fuzzy open sets, Global Journal of Pure and Applied Mathematics, V.13, 2017, pp.251-261.
7. Thakur S.S., Singh S. On fuzzy semi-preopen sets and fuzzy semi-precontinuity, Fuzzy Sets and Systems, V.3, 1998, pp.383-391.
8. Zadeh, L.A. Fuzzy sets, Information and Control, V.8, 1965, pp.338-353.