Proceedings of IAM, V.11, N.1, 2022, pp. 62-67

NUMERICAL METHODS TO SOLVING THE EIGENVALUE PROBLEM FOR THE NONLINEAR SCHREDINGER EQUATION

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Abstract The spectral problem obtained for the nonlinear Schrödinger equation is studied. The initial infinite domain is replaced by the finite one where the grid is constructed. Schrödinger equation on this grid is reduced to the finite-difference equations. The shooting method is used to solve the spectral problem for the obtained finite-difference equation.

Keywords: Schrödinger equation, numerical algorithm, shooting method, spectral problem

AMS Subject Classiffcation: 35Q41, 65Y20, 58C40

1. Introduction

Stable axially and spherically symmetric spatial solitons in a plasma with diatomic ions were studied in [2]. There a new cubic-quadratic nonlinear Schrödinger equation is derived that regulates the dynamics of solitons, and some study of this equation has been carried out. It should be noted that such nonlinear Schrödinger equations occur in the study of various physical processes [1; 3; 7; 8]. Numerous papers are devoted to the study of such equations [5; 10]. In some special cases, one can obtain exact solutions of these equations. However, in the general case, finding the exact solution of such equation is not possible. Therefore, various numerical methods are developed for this purpose [4; 9]. Despite such studies, finding solutions of the nonlinear Schrödinger equation by numerical methods remains to be an important problem.

In this paper, we propose a numerical method for solving the nonlinear Schrödinger equation obtained in [2]. To do this, first the solution of the nonlinear Schrödinger equation with partial derivatives is sought in the form $\psi(x,\tau)=e^{i\lambda\tau}\psi_0(x)$. Then as a result we obtain the ordinary differential Schrödinger equation, which includes the spectral parameter λ. For the numerical solution of the equation obtained, the condition at infinity is replaced by the condition at the end point. Next, dividing the finite interval into a grid and approximating the derivatives of the unknown function by finite differences, we reduce the ordinary differential Schrödinger equation to the finite difference equations. Further, the shooting method is used to solve the spectral problem for the finite difference equations [6].

A numerical algorithm is developed on the base of the proposed method.

2. Problem formulation.

As shown in [2], when considering spherically symmetric plasma oscillations and introducing dimensionless quantities, we obtain the following equation

$$i\frac{\partial\psi}{\partial\tau}+\psi''+\frac{d-1}{x}\psi'-\frac{d-1}{x^2}\psi+(|\psi|^2-|\psi|^4)\psi=0 \qquad (2.1)$$

where $d=2,3$ is a dimension of the space. We require the solution of equation (2.1) satisfy following initial and boundary conditions

$$\psi(x=0;\tau)=\psi(x=\infty;\tau)=0, \qquad (2.2)$$

$$\psi(x;\tau=0)=\psi_0(x), \qquad (2.3)$$

As is shown in [2] the number of plasmans N and Hamiltonian H is defined by the formulas

$$N=\int_0^\infty \Omega_d dx\cdot x^{d-1}|\psi|^2 \ , \qquad (2.4)$$

$$H=\int_0^\infty \Omega_d dx\cdot x^{d-1}\cdot\left\{\left|\frac{1}{x^{d-1}}(x^{d-1}\psi)'\right|^2-\frac{1}{2}|\psi|^4+\frac{1}{3}|\psi|\right\} . \qquad (2.5)$$

Here $\Omega_2=2\pi$, $\Omega_3=4\pi$ are angels in the two and three dimensional spaces, correspondingly.

Solution of equation (2.1) we search in the form $\psi(x;\tau)=e^{i\lambda\tau}\psi_0(x)$. Then it is easy to see that

$$i\frac{\partial\psi}{\partial\tau}=i\cdot i\lambda e^{i\lambda\tau}\psi_0(x)=-\lambda e^{i\lambda\tau}\psi_0(x),$$

$$\psi'(\lambda)=e^{i\lambda\tau}\psi_0'(x),$$

$$\psi''(\lambda)=e^{i\lambda\tau}\psi_0''(x),$$

$$|\psi|^2=|e^{i\lambda\tau}\psi_0(x)|^2=|\psi_0(x)|^2,$$

$$|\psi|^4=|e^{i\lambda\tau}\psi_0(x)|^4=|\psi_0(x)|^4 \, |\psi|^4=|e^{i\lambda\tau}\psi_0(x)|^4=|\psi_0(x)|^4.$$

Considering this from (2.1) we obtain

$$e^{i\lambda\tau}\left[\psi_0''(0)+\frac{d-1}{x}\psi_0'(0)-\lambda\psi_0(x)-\frac{d-1}{x^2}\psi_0(x)+(|\psi_0|^2-|\psi_0|^4)\psi_0(x)\right]=0$$

or

$$\psi_0''(0)+\frac{d-1}{x}\psi_0'(0)-\lambda\psi_0(x)-\frac{d-1}{x^2}\psi_0(x)+(|\psi_0|^2-|\psi_0|^4)\psi_0(x), \quad (2.6)$$

The boundary conditions give

$$\left.\begin{array}{l}\psi_0(0)=0\\ \lim_{x\to\infty}\psi_0(0)=0\end{array}\right\}=0, \quad (2.7)$$

Considering

$$\psi(x;\tau)=e^{i\lambda\tau}\psi_0''(x)$$

we get

$$N(\lambda)=\int_0^\infty \Omega_d dx\cdot x^{d-1}|\psi|^2,$$

$$H(\lambda)=\int_0^\infty \Omega_d dx\cdot x^{d-1}\cdot\left\{\left|\frac{1}{x^{d-1}}(x^{d-1}\psi)'\right|^2-\frac{1}{2}|\psi|^4+\frac{1}{3}|\psi|^6\right\}.$$

We seek the solution for problem (2.6)-(2.7) in the domain $G=\{0\le x\le L\}$ within the condition $\psi_0(L)=0$.

For the numerical solution we introduce the following apporximation

$$\psi_0(x_i)=\psi_0^i,$$

$$\psi_0'(x_i)\approx\frac{\psi_0(x_{i+1})-\psi_0(x_i)}{h}=\frac{\psi_0^{i+1}(x)-\psi_0^{i}(x)}{h},$$

$$\psi_0''(x_i)\approx\frac{\psi_0(x_{i+1})-\psi_0(x_i)+\psi_0(x_{i-1})}{h^2}=\frac{\psi_0^{i+1}-2\psi_0^{i}+\psi_0^{i-1}}{h^2}.$$

Substituting these expressions into equation (2.6) for the points $x=x_i$ we obtain the following finite-difference equation

$$\frac{\psi_0^{i+1}-2\psi_0^{i}+\psi_0^{i-1}}{h^2}+\frac{d-1}{x_i}\frac{\psi_0^{i+1}-\psi_0^{i}}{h}-\lambda\psi_0^i-\frac{d-1}{(x_i)^2}\psi_0^i+(|\psi_0^i|^2-|\psi_0^i|^4)\psi_0^i=0 \quad (2.8)$$

$$\psi_0^{1}=0,\psi_0^{N+1}=0,. \quad (2.9)$$

If in this equation to group the terms relatively ψ_0^i, then we get

$$\psi_0^{i+1}\left[\frac{1}{h^2}+\frac{d-1}{hx_i}\right]+\psi_0^{i}\left[-\frac{2}{h^2}-\frac{d-1}{hx_i}-\frac{d-1}{(x_i)^2}+(|\psi_0^{i}|^2-|\psi_0^{i}|^4)\right]+\psi_0^{i-1}\frac{1}{h^2}=\lambda\psi_0^i\cdot$$

Defining

$$a_i = \left[\frac{1}{h^2} + \frac{d-1}{hx_i} \right],$$

$$b_i = \left[-\frac{2}{h^2} - \frac{d-1}{hx_i} - \frac{d-1}{(x_i)^2} + (\left|\psi_0{}^i\right|^2 - \left|\psi_0{}^i\right|^4) \right],$$

$$c_i = \frac{1}{h^2}$$

from (2.8) one can imply

$$a_i \psi_0{}^{i+1} + b_i \psi_0{}^i + c_i \psi_0{}^{i-1} = \lambda \psi_0{}^i . \quad (2.10)$$

Thus from initial problem (2.1)-(2.3) we obtained spectral problem (2.10), (2.9).

3. Shooting method.

Following [6] as an initial condition we take $\psi_0{}^2 = a$, where a is a parameter and then setting $\lambda = \lambda_0$ solve equation (2.10) with conditions

$$\psi_0{}^1 = 0, \psi_0{}^2 = a \quad , \quad . \quad (3.1)$$

From (2.10) define

$$\psi_0{}^{i+1} = \frac{1}{a_i} \left[\lambda_0 \psi_0{}^{i+1} + b_i \psi_0{}^i + c_i \psi_0{}^{i-1} \right] . \quad (3.2)$$

Using formula (3.2) consistently determine the values $\psi_0{}^2$, $\psi_0{}^3$, ... , $\psi_0{}^{N+1}$. Obviously $\psi_0{}^{N+1} = \psi_0{}^{N+1}(\lambda_0)$ i.e. $\psi_0{}^{N+1}$ depends on λ_0. If $\psi_0{}^{N+1}(\lambda_0) = 0$ then it means that $\lambda = \lambda_0$ is an eigenvalue of problem (2.10), (2.9). Generally

$$\psi_0{}^{N+1}(\lambda_0) \neq 0 .$$

Then you need to vary λ in such a way that the equality

$$\psi_0{}^{N+1}(\lambda) \approx 0$$

hold with some exactness. Thus solution of spectral problem (2.10), (2.9) is reduced to the finding of the roots of the nonlinear equation

$$\psi_0{}^{N+1}(\lambda) = 0 . \quad (3.3)$$

Note that for the solution of equation (3.3) different methods may be used. Suppose that $\psi_0{}^{N+1}(\lambda_0) = b > 0$. The we find $\lambda_1 > \lambda_0$ such that $\psi_0{}^{N+1}(\lambda_1) = c < 0$. Then in the interval (λ_0, λ_1) there exists at least one solution of equation (3.3) and we seek the rot in this interval. To find the root of equation (3.3) we apply the simple method- the method of divisionby 2. First we find these values λ_0, λ_1 such that provide fulfilment of the conditions $\psi_0^{N+1}(\lambda_0)\cdot\psi_0^{N+1}(\lambda_1)<0$. Then from the formula

$$\lambda_{i+2}=\frac{\lambda_i+\lambda_{i+1}}{2}$$

we determine the value λ_2. Then we check the condition $\psi_0^{N+1}(\lambda_2)=0$. If this condition is not satified we check the condition

$$\psi_0^{N+1}(\lambda_0)\cdot\psi_0^{N+1}(\lambda_2)<0 \ .$$

If this condition take place we seek the root of equation (3.3) in the interval (λ_0,λ_2), otherwise in the interval (λ_2,λ_1) . Iteratively continuing this process we will find the values $\lambda_3,\lambda_4,...,\lambda_k$. The process continues until is executed

$$\psi_0^{N+1}(\lambda_k)<\varepsilon,$$

where ε is a requared ecatness. The obtained by this way value $\lambda=\lambda_k$ will be approximate eigenvalue. Then the corresponding eigenfunction may be found.

4. **Conclusion**. Thus, the paper proposes a numerical method for solving the nonlinear Schrödinger equation. First, the solution of the nonlinear Schrödinger equation with partial derivatives is sought in a special form, as a result of which an ordinary differential Schrödinger equation is obtained, which includes a spectral parameter. For the numerical solution of the resulting equation, the condition at infinity is replaced by the condition at the end point. Further, the Schrödinger ordinary differential equation is reduced to finite-difference equations, for the solution of which the shooting method is used.

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