Proceedings of IAM, V.11, N.1, 2022 pp.27-38

NOVEL ESTIMATIONS FOR QUASI-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATORS WITH STRONG KERNELS

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Abstract: New and more general inequalities can be obtained by using non-singular and non-local integral operators with different kinds of convex functions. In this study, we obtained novel integral inequalities for quasi-convex functions which is different kind of convex functions with the help of Atangana-Baleanu fractional integral operator by using the integral identity that has been proved by Set et al. in [7]. Main findings produce new estimations for particular values of the order of the fractional integral operator.

Keywords: Convex functions, quasi-convex functions, Hölder inequality, Young inequality, power mean inequality, Atangana-Baleanu fractional integral operators.

AMS Subject Classification: 26A33, 26A51, 26D10.

1 Introduction

Different kinds of functions that have useful features are used in many areas of mathematics as analysis, geometry, applied mathematics, fractional analysis. One of these kinds of functions is convex functions. This interesting class of function is defined as follow.

Definition 1.1 *The function $f:[a,b]\subseteq\mathbb{R}\rightarrow\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y) \tag{1}$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

After by the defining of convex function, mathematicians obtained many new inequalities by using this valuable concept. The most important one of these inequalities has been given in the following inequality, which is well-known in the literature and proved by Hermite-Hadamard.

Assume that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.
We will remember the definition of quasi-convexity as following (see, e.g. [12]):

Definition 1.2 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ for all $\lambda \in [0,1]$ and all $x, y \in I$, if the following inequality*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

holds, then f is called a quasi-convex function on I.

To provide detail information on convexity, some different type inequalities and more, see the papers [5,7,9,14-16,20,22-25].

The fractional analysis, which is the theory of derivatives and integrals of fractional order, defines a wide variety of different types of operators with non-integer order. Fractional analysis let us to define various phenomena and effects in natural and social sciences and it is applied in diverse and widespread fields of engineering and science such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics, and signals processing. Fractional analysis realizes its effectiveness and applications in these areas with the help of different fractional operators defined by many mathematicians. Now let's introduce some fractional derivative and integral operators that have proven their effectiveness in different fields.

Firstly, we recall the Caputo-Fabrizio fractional integral operators as following.

Definition 1.3 *[2] Let $f \in H^1(0,b)$, $b > a$, $\alpha \in 0,1]$ then, the definition of the left and right sides of Caputo-Fabrizio fractional integral is:*

$$\left({}^{CF}_{\ a}I^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_a^t f(y)dy,$$

and

$$\left({}^{CF}I^{\alpha}_{b}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_t^b f(y)dy,$$

where $B(\alpha)$ is normalization function.
In this paper, we will denote normalization function as $B(\alpha)$ and $\Gamma(.)$ is Gamma function.

Atangana and Baleanu introduced a new derivative operator using Mittag-Leffler function in Caputo-Fabrizio derivative operator to eliminate some of the shortcomings of the Caputo-Fabrizio derivative operator as following.

Definition 1.4 *[8] Let* $f \in H^1(a,b)$, $b > a$, $\alpha \in 0,1]$ *then, the definition of the new fractional derivative is given:*

$$ {}^{ABC}_{\ a}D^{\alpha}_{t}[f(t)] = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{(1-\alpha)}\right]dx. $$

When alpha is zero in (1.3), the original function is not recovered except when at the origin the function vanishes. To avoid this issue, Atangana and Baleanu propose the following definition.

Definition 1.5 *[8] Let* $f \in H^1(a,b)$, $b > a$, $\alpha \in 0,1]$ *then, the definition of the new fractional derivative is given:*

$$ {}^{ABR}_{\ a}D^{\alpha}_{t}[f(t)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{(1-\alpha)}\right]dx. $$

Equations (1.3) and (1.4) have a non-local kernel. Also in equation (1.3) when the function is constant we get zero.
After Atangana-Baleanu defined the derivative operators above, they defined the following associated integral operator, the definition of which has become a necessity.

Definition 1.6 *[8] The fractional integral associate to the new fractional derivative with non-local kernel of a function* $f \in H^1(a,b)$ *as defined:*

$$ {}^{AB}_{\ a}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy\,, $$

where $b > a, \alpha \in [0,1]$.
In [1], Abdeljawad and Baleanu introduced right hand side of integral operator as following; The right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$ {}^{AB}I^{\alpha}_{b}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy. $$

For more information related to different kinds of fractional operators, we recommend to the readers the following papers [2-4,6,10,11,13,17-19,21].
We will give the identity that will be useful to obtain main results and provided by Set et al. in [7] as follow.
Lemma 1.1 *Let* $f:[a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *with* $a < b$. *Then, we have the following identity for Atangana-Baleanu fractional integral operators*

$$ \frac{2(b-a)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right] $$
$$ -\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a) + {}^{AB}_{\ a}I^{\alpha}f\left(\frac{a+b}{2}\right) + {}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b) + {}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right] $$

$$= \int_0^1 \left((1-t)^\alpha - t^\alpha\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt + \int_0^1 \left(t^\alpha - (1-t)^\alpha\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt,$$

where $\alpha, t \in [0,1]$, $\Gamma(.)$ is Gamma function and $B(\alpha)$ is a normalization function.

The purpose of this paper is to provide some new integral inequalities for differentiable quasi-convex functions that includes the Atangana-Baleanu integral operator. Some special cases are also considered.

2 Main Results

Theorem 2.1 *Let $f: [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$ and $f' \in L_1[a,b]$. If $|f'|$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\left| \frac{2(b-a)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[{}^{AB}I^\alpha_{\frac{a+b}{2}} f(a) + {}^{AB}_{\ a}I^\alpha f\left(\frac{a+b}{2}\right) + {}^{AB}_{\frac{a+b}{2}}I^\alpha f(b) + {}^{AB}I^\alpha_b f\left(\frac{a+b}{2}\right) \right] \right|$$

$$\leq \frac{4\max\{|f'(a)|, |f'(b)|\}}{\alpha+1},$$

where $\alpha \in [0,1]$, $B(\alpha)$ is the normalization function.

Proof. By using Lemma 1.1, we can write

$$\left| \frac{2(b-a)^\alpha + (1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] \right.$$

$$\left. - \frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}} \left[{}^{AB}I^\alpha_{\frac{a+b}{2}} f(a) + {}^{AB}_{\ a}I^\alpha f\left(\frac{a+b}{2}\right) + {}^{AB}_{\frac{a+b}{2}}I^\alpha f(b) + {}^{AB}I^\alpha_b f\left(\frac{a+b}{2}\right) \right] \right|$$

$$= \left| \int_0^1 \left((1-t)^\alpha - t^\alpha\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt + \int_0^1 \left(t^\alpha - (1-t)^\alpha\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \right|$$

$$\leq \int_0^1 (1-t)^\alpha \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t^\alpha \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt$$

$$+\int_0^1 t^\alpha \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt+\int_0^1 (1-t)^\alpha \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt.$$

By using the definition of quasi-convexity for $|f'|$, we get

$$\left|\frac{2(b-a)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^\alpha_{\frac{a+b}{2}}f(a)+{}^{AB}_{\ a}I^\alpha f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^\alpha f(b)+{}^{AB}I^\alpha_b f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \int_0^1 (1-t)^\alpha \max\{|f'(a)|,|f'(b)|\}dt+\int_0^1 t^\alpha \max\{|f'(a)|,|f'(b)|\}dt$$

$$+\int_0^1 t^\alpha \max\{|f'(b)|,|f'(a)|\}dt+\int_0^1 (1-t)^\alpha \max\{|f'(b)|,|f'(a)|\}dt.$$

By computing the above integrals, we obtain

$$\left|\frac{2(b-a)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^\alpha_{\frac{a+b}{2}}f(a)+{}^{AB}_{\ a}I^\alpha f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^\alpha f(b)+{}^{AB}I^\alpha_b f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{4\max\{|f'(a)|,|f'(b)|\}}{\alpha+1}$$

and the proof is completed.

Corollary 2.1 *In Theorem 2.1, if we choose* $\alpha=1$, *we obtain*

$$\left|\frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{b-a}-\frac{4}{(b-a)^2}\int_a^b f(x)dx\right|\leq \max\{|f'(a)|,|f'(b)|\}.$$

Theorem 2.2 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f'\in L_1[a,b]$. *If* $|f'|^q$ *is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\left|\frac{2(b-a)^\alpha+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{\ a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$
$$\leq \frac{4\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}}.$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, $B(\alpha)$ is the normalization function.

Proof. By using the identity that is given in Lemma 1.1, we have

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{\ a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$

$$\leq \int_0^1 (1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt$$

$$+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt+\int_0^1 (1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt.$$

By applying Hölder inequality, we have

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{\ a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$
$$\leq\left(\int_0^1 (1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}.$$

By using quasi-convexity of $|f'|^q$, we obtain

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$

$$\leq\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\max\{|f'(b)|^q,|f'(a)|^q\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\max\{|f'(b)|^q,|f'(a)|^q\}dt\right)^{\frac{1}{q}}.$$

By calculating the integrals that is in the above inequalities, we get

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$

$$\leq\frac{4\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}}.$$

This completes the proof.

Corollary 2.2 *In Theorem 2.2, if we choose* $\alpha=1$, *we obtain*

$$|\frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{b-a}-\frac{4}{(b-a)^2}\int_a^b f(x)dx|\leq\frac{2\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}}.$$

Theorem 2.3 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f'\in L_1[a,b]$. *If* $|f'|^q$ *is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$

$$\leq 4\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|f'(a)|^q,|f'(b)|^q\}}{\alpha+1}\right)^{\frac{1}{q}},$$

where $\alpha\in[0,1]$, $q\geq 1$, $B(\alpha)$ is the normalization function.

Proof. By using Lemma 1.1, we get

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt$$

$$+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt+\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt.$$

By applying power-mean inequality, we obtain

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}.$$

By using the definition of quasi-convexity for $|f'|^q$, we deduce

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$

$$\leq\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\max\{|f'(b)|^q,|f'(a)|^q\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\max\{|f'(b)|^q,|f'(a)|^q\}dt\right)^{\frac{1}{q}}$$
$$=4\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|f'(a)|^q,|f'(b)|^q\}}{\alpha+1}\right)^{\frac{1}{q}}.$$

So, the proof is completed.

Corollary 2.3 *In Theorem 2.3, if we choose* $\alpha=1$, *we obtain*

$$|\frac{\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]}{b-a}-\frac{4}{(b-a)^2}\int_a^b f(x)dx|\leq 2^{\frac{1}{q}}\left(\frac{\max\{|f'(a)|^q,|f'(b)|^q\}}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.4 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f'\in L_1[a,b]$. *If* $|f'|^q$ *is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]$$

$$-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]|$$
$$\leq\frac{4}{p(\alpha p+1)}+\frac{4\max\{|f'(a)|^q,|f'(b)|^q\}}{q},$$

where $p^{-1}+q^{-1}=1$, $\alpha\in[0,1]$, $q>1$, $B(\alpha)$ is a normalization function.

Proof. By using identity that is given in Lemma 1.1, we get

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt$$

$$+\int_0^1 t^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt+\int_0^1(1-t)^{\alpha}\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt.$$

By using the Young inequality as $xy\leq\frac{1}{p}x^p+\frac{1}{q}y^q$, we can write

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\begin{aligned}&\leq\frac{1}{p}\int_0^1(1-t)^{\alpha p}dt+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\\&+\frac{1}{p}\int_0^1 t^{\alpha p}dt+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\\&+\frac{1}{p}\int_0^1 t^{\alpha p}dt+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\\&+\frac{1}{p}\int_0^1(1-t)^{\alpha p}dt+\frac{1}{q}\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt.\end{aligned}$$

By using quasi-convexity of $|f'|^q$, we obtain

$$\left|\frac{2(b-a)^{\alpha}+(1-\alpha)2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]\right.$$

$$\left.-\frac{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[{}^{AB}I^{\alpha}_{\frac{a+b}{2}}f(a)+{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}f(b)+{}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\begin{aligned}&\leq\frac{1}{p}\int_0^1(1-t)^{\alpha p}dt+\frac{1}{q}\int_0^1\max\{|f'(a)|^q,|f'(b)|^q\}dt\\&+\frac{1}{p}\int_0^1 t^{\alpha p}dt+\frac{1}{q}\int_0^1\max\{|f'(a)|^q,|f'(b)|^q\}dt\\&+\frac{1}{p}\int_0^1 t^{\alpha p}dt+\frac{1}{q}\int_0^1\max\{|f'(b)|^q,|f'(a)|^q\}dt\\&+\frac{1}{p}\int_0^1(1-t)^{\alpha p}dt+\frac{1}{q}\int_0^1\max\{|f'(b)|^q,|f'(a)|^q\}dt\\&=\frac{4}{p(\alpha p+1)}+\frac{4\max\left\{|f'(a)|^q,|f'(b)|^q\right\}}{q}.\end{aligned}$$

So, the proof is completed.

Corollary 2.4 *In Theorem 2.4, if we choose* $\alpha = 1$, *we obtain*

$$\left|\frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{b-a} - \frac{4}{(b-a)^2}\int_a^b f(x)dx\right| \leq \frac{2}{p^2+p} + \frac{2\max\left\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\right\}}{q}.$$

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