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DIFFERENTIAL SUBORDINATION APPLICATIONS TO A CLASS OF $p-$ VALENT FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTION

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Abstract. The purpose of the present paper is to introduce suitable differential subordination preserving properties for certain subclasses of functions which are analytic and p-valent in the open unit disc defined by generalized Mittag-Leffler function that is entire in the complex z-plane, by making use of the subordination and the classical Hadamard product definitions.

Keywords: analytic, p - valent functions, Hadamard product, subordination and Mittag-Leffler function.
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1. Introduction.

$$f(z)=z^p+\sum_{n=m}^{\infty} a_{n+p}z^{n+p} \ (p,m\in N=\{1,2,\ldots\}; z\in U=\{z\in \mathrm{C} \text{ and } |z|<1\}). \qquad (1)$$

Denote by $A_p(m)$ the class of analytic $p-$ valent functions of the form:
We note that : $A_p(1)=A_p$.
For two functions $f(z)$ and $g(z)$, analytic in U, $f(z)$ is subordinate to $g(z)(f(z)\prec g(z))$ in U, if there exists a function $\omega(z)$, analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$, $f(z)=g(\omega(z))$ $(z\in U)$ and if $g(z)$ is univalent in U, then (see for details [1], [5] and also [7]):

$$f(z)\prec g(z) \Leftrightarrow f(0)=g(0) \text{ and } f(\mathrm{U})\subset g(\mathrm{U}).$$

The Hadamard product of $f(z)$ and $g(z)$ given by $g(z)=z^p+\sum_{n=m}^{\infty} b_{n+p}z^{n+p}$, is defined by

$$(f*g)(z)=z^p+\sum_{n=m}^{\infty} a_{n+p}b_{n+p}z^{n+p} = (g*f)(z). \qquad (2)$$

The Mittag-Leffler function $E_\alpha(z)$ ($z \in C$) ([10] and [11]) is defined by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}z^n (\alpha\in C, Re(\alpha)>0).$$

For $\alpha,\beta,\gamma \in C$, $Re(\alpha) > max\ \{0, Re(k)-1\}$ and $Re(k) > 0$, Srivastava and Tomovski [18] generalized Mittag-Leffler function by the function

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\Gamma(n\alpha+\beta)n!} z^n \tag{3}$$

and proved that it is an entire function in the complex $z-$ plane, where

$$(\gamma)_\theta = \frac{\Gamma(\gamma+\theta)}{\Gamma(\gamma)} \begin{cases} 1, & \theta = 0 \\ \gamma(\gamma+1)\ldots(\gamma+\theta-1), & \theta \neq 0 \end{cases}.$$

Aouf and Seoudy [3], used the function $E_{\alpha,\beta}^{\gamma,k}(z)$ and defined the $p-$ valent function

$$Q_{p,\alpha,\beta}^{\gamma,k}(z) = z^p\Gamma(\beta)E_{\alpha,\beta}^{\gamma,k}(z) = z^p + \sum_{n=m}^{\infty} \frac{\Gamma(\beta)\Gamma(\gamma+nk)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)n!} z^{p+n},$$

$$(Re\,\alpha = 0 \text{ when } Re\,k = 1 \text{ with } \beta \neq 0), \tag{4}$$

and for $f(z) \in A_p(m)$, they defined the operator

$$H_{p,\alpha,\beta}^{\gamma,k} f(z) = Q_{p,\alpha,\beta}^{\gamma,k}(z) * f(z)$$

$$= z^p + \sum_{n=m}^{\infty} \frac{\Gamma(\beta)\Gamma(\gamma+nk)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)n!} a_{p+n} z^{p+n}. \tag{5}$$

From (5) it is easy to have

$$kz(H_{p,\alpha,\beta}^{\gamma,k} f(z))' = \gamma H_{p,\alpha,\beta}^{\gamma+1,k} f(z) - (\gamma - pk)H_{p,\alpha,\beta}^{\gamma,k} f(z)\ (k > 0) \tag{6}$$

and

$$\alpha z\left(H_{p,\alpha,\beta+1}^{\gamma,k} f(z)\right)' = \beta\ H_{p,\alpha,\beta}^{\gamma,k} f(z) - (\beta - p\alpha)\ H_{p,\alpha,\beta+1}^{\gamma,k} f(z), \alpha \neq 0. \tag{7}$$

We note that:

(i) $H_{p,0,\beta}^{1,1} f(z) = f(z)$;

(ii) $H_{p,0,\beta}^{2,1} f(z) = (1-p)f(z) + zf'(z)$;

(iii) $\mathbf{H}_{1,0,\beta}^{2,1} f(z) = zf'(z)$;

(iv) $H_{p,0,1}^{1,1}(\frac{z^p}{1-z}) = z^p e^z$;

(v) $H_{1,0,1}^{1,1}(\frac{z}{1-z}) = ze^z$.

Using the operator $H_{p,\alpha,\beta}^{\gamma,k} f(z)$, we have the following definition.

Definition 1. For fixed A and $B(-1 \leq B < A \leq 1)$, we say that a function $f \in A_p(m)$ is in the class $S_{p,m}^{\gamma,k}(\alpha,\beta;A,B)$, if it satisfies:

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{8}$$

In view of the definition of differential subordination, (8) is equivalent to:

$$\left|\frac{\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{p-1}}-p}{B\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{p-1}}-pA}\right| < 1. \tag{9}$$

We note that:

(i)

$$S_{p,1}^{1,1}(0,1;A,B) = S_p(A,B)\ (-1 \leq B < A \leq 1; z \in U)$$
$$= \left\{f \in A_p\ :\ \frac{f'(z)}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}\right\},$$

the class $S_p(A,B)$ was introduced and studied by Chen [9].

(ii)

$$S_{p,m}^{\gamma,k}\left(\alpha,\beta;1-\frac{2\eta}{p},-1\right) = S_{p,m}^{\gamma,k}(\alpha,\beta,\eta)(0 \leq \eta < p)$$
$$= \left\{f \in A_p(m)\ :\ Re\left\{\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{z^{p-1}}\right\} > \eta\right\}. \tag{10}$$

2. Preliminary results.

The following lemmas will be required in our investigation.

Lemma 1[7,8] . Let $h(z)$ be analytic and convex (univalent) function in U with $h(0) = 1$. Also let

$$\varphi(z) = 1 + d_m z^m + d_{m+1}z^{m+1} + \dots, \tag{11}$$

be analytic in U. If

$$\varphi(z) + \frac{z\varphi'(z)}{\tau} \prec h(z)(Re(\tau) \geq 0; \tau \neq 0; z \in U), \tag{12}$$

then

$$\varphi(z) \prec \Psi(z) = \frac{\tau}{m}z^{-\frac{\tau}{m}}\int_0^z t^{\frac{\tau}{m}-1}h(t)dt \prec h(z), \tag{13}$$

and Ψ is the best dominant of (12).

Lemma 2 [20]**.** Let μ be a positive measure on the unit interval $[0,1]$. Let $g(z,t)$ be a complex valued function defined on $U \times [0,1]$ such that $g(.,t)$ is analytic in U for each $t \in [0,1]$ and such that $g(z,.)$ is μ integrable on $[0,1]$ for all $z \in U$. In addition, suppose that $Re\{g(z,t)\} > 0$, $g(-r,t)$ is real and

$$Re\left\{\frac{1}{g(z,t)}\right\} \geq \frac{1}{g(-r,t)} \ (|z| \leq r < 1; t \in [0,1]).$$

If the function G is defined by

$$G(z) = \int_0^1 \quad g(z,t)d\mu(t),$$

then

$$Re\left\{\frac{1}{G(z)}\right\} \geq \frac{1}{G(-r)} \ (|z| \leq r < 1).$$

Each of the identities (asserted by Lemma 3) is fairly well known (cf., e.g., [19]).

Lemma 3 [19]**.** For real or complex numbers $a,$ b and c $(c \neq 0,-1,-2,\dots)$

$$\int_0^1 \quad t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \ {}_2F_1(a,b;c;z)(Re(c) < Re(b) > 0); \tag{14}$$

$$2F_1(a,b;c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)(z \neq 1) \tag{15}$$

and

$$2F_1(a,b;c;z) = \ {}_2F_1(b,a;c;z). \tag{16}$$

Lemma 4 [17]**.** Let Φ be analytic in U with

$$\Phi(0) = 1 \text{ and } Re\{\Phi(z)\} > \frac{1}{2}.$$

Then, for any function F analytic in $U,$ $(\Phi * F)(U)$ is contained in the convex hull of $F(U)$.

Lemma 5 [14]Let φ be analytic in U with $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $0 < |z| < 1,$ and let $A, B \in C$ with $A \neq B, |B| \leq 1.$

(i) Let $B \neq 0$ and $\lambda \in C\backslash\{0\}$ satisfy either

$$\left|\frac{\lambda(A-B)}{B} - 1\right| \leq 1 \text{ or } \left|\frac{\lambda(A-B)}{B} + 1\right| \leq 1.$$

If φ satisfies

$$1 + \frac{z\varphi'(z)}{\lambda\varphi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\varphi(z) \prec (1+Bz)^{\lambda\left(\frac{A-B}{B}\right)}$$

and this is the best dominant.

(ii) Let $B = 0$ and $\lambda \in C\backslash\{0\}$ be such that $|\lambda A| < \pi.$ If φ satisfies

$$1 + \frac{z\varphi'(z)}{\lambda\varphi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\varphi(z) \prec e^{\lambda A z}$$

and this is the best dominant.
We used the technique used by ([2,4]and [12]).

3. Main Inclusion Relationships.

Unless otherwise mentioned, we assume throughout this paper that $-1 \leq B < A \leq 1, \alpha, \beta, \gamma \in C,\ Re(\alpha) > max\ \{0, Re(k) - 1\}$ and $Re(k) > 0, \delta > 0, f(z)$ given by (1.1) and $z \in U$.

Theorem 1. Let $\gamma \neq 0$ the function $f(z)$ satisfy:

$$(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \tag{17}$$

then

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{pz^{p-1}} \prec \Psi(z) \prec \frac{1+Az}{1+Bz}, \tag{18}$$

where

$$\Psi(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1; \frac{\gamma}{\delta k m} + 1; \frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{\gamma}{\gamma + \delta k m} Az, & B = 0. \end{cases} \tag{19}$$

is the best dominant of (19). Furthermore,

$$Re\left\{\frac{\left(H_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{pz^{p-1}}\right\} > \rho (0 \leq \rho < 1), \tag{20}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1; \frac{\gamma}{\delta k m} + 1; \frac{B}{B-1}\right), & B \neq 0 \\ 1 - \frac{\gamma}{\gamma + \delta k m} A, & B = 0. \end{cases} \tag{21}$$

The result is the best possible.
Proof Let

$$\varphi(z) = \frac{\left(H_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{pz^{p-1}}, \tag{22}$$

where φ is given by (11). Differentiating (22) and using (6), we get

$$(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k} f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{pz^{p-1}} = \varphi(z) + \frac{\delta k z \varphi'(z)}{\gamma} \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 1 for $\tau = \frac{\gamma}{\delta k}$, we get

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} \prec \Psi(z) = \frac{\gamma}{\delta km} z^{-\frac{\gamma}{\delta km}} \int_0^z t^{\frac{\gamma}{\delta km}-1}\left(\frac{1+At}{1+Bt}\right)dt$$
$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\gamma}{\delta km}+1;\frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{\gamma}{\gamma+\delta km}Az, & B = 0. \end{cases}$$

This proves (18) of Theorem 1. In order to prove (19), we need to show that

$$\inf_{|z|<1}\{Re(\Psi(z))\} = \Psi(-1). \qquad (23)$$

We have

$$Re\left\{\frac{1+Az}{1+Bz}\right\} \geq \frac{1-Ar}{1-Br} \; (|z| \leq r < 1).$$

Putting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \text{ and } dv(\zeta) = \frac{\gamma}{\delta km}\zeta^{\frac{\gamma}{\delta km}-1}d\zeta(0 \leq \zeta \leq 1),$$

which is a positive measure on $[0,1]$, we obtain

$$\Psi(z) = \int_0^1 G(z,\zeta)dv(\zeta).$$

Then

$$Re(\Psi(z)) \geq \int_0^1 \frac{1-A\zeta r}{1-B\zeta r}dv(\zeta) = \Psi(-r) \; (|z| \leq r < 1).$$

Assuming $r \to 1^-$ in the above inequality, we obtain (23). The result in (20) is the best possible and Ψ is the best dominant of (18). This completes the proof of Theorem 1.

Taking $\delta = 1$ in Theorem 1, we obtain

Corollary1. The following inclusion relation holds:

$$S_{p,m}^{\gamma+1,k}(\alpha,\beta;A,B) \subset S_{p,m}^{\gamma,k}\left(\alpha,\beta;1-\frac{2\eta}{p},-1\right)$$
$$\subset S_{p,m}^{\gamma,k}(\alpha,\beta;A,B)(0 \leq \eta < p),$$

where

$$\eta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\gamma}{km}+1;\frac{B}{B-1}\right), & B \neq 0 \\ 1 - \frac{\gamma}{\gamma+km}A, & B = 0. \end{cases}$$

The result is the best possible.

Taking $\delta = 1, A = 1 - \frac{2\eta}{p}(0 \leq \eta < p)$ and $B = -1$ in Theorem 1, we obtain

Corollary2. The following inclusion relation holds:

$$S_{p,m}^{\gamma+1,k}(\alpha,\beta;\eta) \subset S_{p,m}^{\gamma,k}\big(\alpha,\beta;\zeta(m,\gamma,k,\eta)\big) \subset S_{p,m}^{\gamma,k}(\alpha,\beta;\eta),$$

where

$$\zeta(m,\gamma,k,\eta) = \eta + (p-\eta)\left\{ 2F_1\left(1,1;\frac{\gamma}{km}+1;\frac{1}{2}\right)\right\}.$$

The result is the best possible.

Theorem 2. Let $f(z) \in S_{p,m}^{\gamma,k}(\alpha,\beta,\eta)(0 \le \eta < p)$, then

$$Re\left\{(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{pz^{p-1}}\right\} > \eta \ (|z| < R), \tag{24}$$

where

$$R = \left\{\frac{\sqrt{k^2\delta^2m^2+\gamma^2}-k\delta m}{\gamma}\right\}^{\frac{1}{m}}. \tag{25}$$

The result is the best possible.

Proof Since $f(z) \in S_{p,m}^{\gamma,k}(\alpha,\beta,\eta)$, let

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)u(z), \tag{26}$$

where $u(z)$ is given by (11) and $Re\{u(z)\} > 0$. Differentiating (26) and using (6), we get

$$\frac{1}{1-\eta}\left[(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{pz^{p-1}} - \eta\right] = u(z) + \frac{k\delta zu'(z)}{\gamma}. \tag{27}$$

Applying the following estimate [6]:

$$\frac{|zu'(z)|}{Re\{u(z)\}} \le \frac{2mr^m}{1-r^{2m}} \ (|z| = r < 1),$$

in (27), we get

$$\frac{1}{1-\eta}Re\left[(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} + \delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{pz^{p-1}} - \eta\right]$$

$$\ge Re(u(z))\left(1 - \frac{2k\delta mr^m}{\gamma(1-r^{2m}))}\right). \tag{28}$$

It is easily seen that the right-hand side of (28) is positive, if $r < R$, where R is given by (25).

In order to show that the bound R is the best possible, we consider the function $f \in A_p(m)$ defined by

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)\frac{1+z^m}{1-z^m}.$$

Noting that

$$\frac{1}{1-\eta}\left[(1-\delta)\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}}+\delta\frac{\left(H_{p,\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{pz^{p-1}}-\eta\right]$$

$$=\frac{\gamma(1-z^{2m})+2k\delta mz^{m}}{\gamma(1-z^{m})^{2}}=0,$$

for

$$z=R\ exp\left\{\frac{i\pi}{m}\right\}.$$

This completes the proof of Theorem 2.

Putting $\delta=1$ in Theorem 2, we obtain the following result.

Corollary 3. If $f(z)\ \in\ S_{p,m}^{\gamma,k}(\alpha,\beta,\eta)(0\leq\eta<p)$, then $f(z)\ \in$ $S_{p,m}^{\gamma+1,k}(\alpha,\beta,\eta)$ for $|z|<R^{*}$, where

$$R^{*}=\left\{\frac{\sqrt{k^{2}m^{2}+\gamma^{2}}-km}{\gamma}\right\}^{\frac{1}{m}}.$$

The result is the best possible.

For $c>-p$ and $f(z)\in A_{p}(m)$, the integral operator $J_{c,p}f(z)$: $A_{p}(m)\to$ $A_{p}(m)$ is defined by

$$J_{c,p}f(z)=\frac{c+p}{z^{c}}\int_{0}^{z}\ t^{c-1}f(t)dt$$

$$=\left(z^{p}+\sum_{k=1}^{\infty}\ \frac{c+p}{c+p+k}z^{p+k}\right)*f(z).$$

$$=z^{p}{}_{2}F_{1}(1,c+p,c+p+1;z)*f(z). \tag{29}$$

The operator $J_{c,p}f(z)$ was introduced by Saitoh [15] and Saitoh et al. [16].

From (29), we get

$$z\left(H_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z)\right)'=(c+p)H_{p,\alpha,\beta}^{\gamma,c}f(z)-cH_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z). \tag{30}$$

Theorem 3. Let $f(z)\in\ S_{p,m}^{\gamma,k}(\alpha,\beta;A,B)$ and $J_{c,p}$ defined by (29). Then

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z)\right)'}{pz^{p-1}}\prec\Theta(z)\prec\frac{1+Az}{1+Bz}, \tag{31}$$

where

$$\Theta(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_{2}F_{1}\left(1,1;\frac{c+p}{m}+1;\frac{Bz}{1+Bz}\right), & B\neq0\\ 1+\frac{c+p}{c+p+m}Az, & B=0.\end{cases} \tag{32}$$

is the best dominant of (32). Furthermore,

$$Re\left\{\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z)\right)'}{pz^{p-1}}\right\} > \sigma \; (0 \le \sigma < 1), \tag{33}$$

where

$$\sigma = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{c+p}{m}+1;\frac{B}{B-1}\right), & B \ne 0 \\ 1 - \frac{c+p}{c+p+m}A, & B = 0. \end{cases} \tag{34}$$

The result is the best possible.

Proof Let

$$\Theta(z) = \frac{\left(H_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z)\right)'}{pz^{p-1}}, \tag{35}$$

where Θ is given by (11). Differentiating (35) and using (30), we get

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}J_{c,p}f(z)\right)'}{pz^{p-1}} = \Theta(z) + \frac{z}{(c+p)}\Theta'(z) \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by using the technique used in proving Theorem 1.

Theorem 4. Let $f(z)$ be in the class $A_p(m)$. Also let $g(z) \in A_p(m)$ satisfying:

$$Re\left\{\frac{H_{p,\alpha,\beta}^{\gamma,k}g(z)}{z^p}\right\} > 0.$$

If

$$\left|\frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{H_{p,\alpha,\beta}^{\gamma,k}g(z)} - 1\right| < 1,$$

then

$$Re\left\{\frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}f(z)}\right\} > 0 \; (|z| < R_0), \tag{36}$$

where

$$R_0 = \frac{\sqrt{9m^2+4p(p+m)} - 3m}{2(p+m)}. \tag{37}$$

Proof Let

$$\varphi(z) = \frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{H_{p,\alpha,\beta}^{\gamma,k}g(z)} - 1 = e_m z^m + e_{m+1}z^{m+1} + \ldots, \tag{38}$$

we note that φ is analytic in U, with $\varphi(0) = 0$ and $|\varphi(z)| \le |z|^m$. Then, by applying the familiar Schwarz Lemma [13], we have $\varphi(z) = z^m\Psi(z)$ is analytic in U and $|\Psi(z)| \le 1 \quad (z \in U)$. Therefore, (38) leads to

$$H_{p,\alpha,\beta}^{\gamma,k}f(z) = H_{p,\alpha,\beta}^{\gamma,k}g(z)(z^m\Psi(z)+1). \tag{39}$$

Differentiating (39) logarithmically with respect to z , we obtain

$$\frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}f(z)} = \frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}g(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}g(z)} + \frac{z^m[m\Psi(z)+z\Psi'(z)]}{1+z^m\Psi(z)}. \tag{40}$$

Letting $\chi(z) = \frac{H_{p,\alpha,\beta}^{\gamma,k}g(z)}{z^p}$, we see that the function χ is of the form (11), is analytic in U , $Re\,\chi(z) > 0$ and

$$\frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}g(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}g(z)} = \frac{z\chi'(z)}{\chi(z)} + p,$$

so, we find from (38) that

$$Re\left\{\frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}f(z)}\right\} \geq p - \left|\frac{z\chi'(z)}{\chi(z)}\right| - \left|\frac{z^m[m\Psi(z)+z\Psi'(z)]}{1+z^m\Psi(z)}\right|. \tag{41}$$

Using the following known estimates [6] (see also [13]):

$$\left|\frac{\chi'(z)}{\chi(z)}\right| \leq \frac{2mr^{m-1}}{1-r^{2m}} \text{ and } \left|\frac{m\Psi(z)+z\Psi'(z)}{1+z^m\Psi(z)}\right| \leq \frac{m}{1-r^m}\,(|z| = r < 1),$$

in (41), we have

$$Re\left\{\frac{z\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{H_{p,\alpha,\beta}^{\gamma,k}f(z)}\right\} \geq \frac{p-3mr^m-(p+m)r^{2m}}{1-r^{2m}},$$

which is certainly positive, provided that $r < R_0$, R_0 given by (37).

Theorem 5. Let the function $f(z) \in S_{p,m}^{\gamma,k}(\alpha,\beta;A,B)$ and $g(z) \in A_p(m)$ satisfying:

$$Re\,\frac{g(z)}{z^p} > \frac{1}{2}.$$

Then

$$(f * g) \in S_{p,m}^{\gamma,k}(\alpha,\beta;A,B).$$

Proof We have

$$\frac{\left(H_{p,\alpha,\beta}^{\gamma,k}(f*g)(z)\right)'}{pz^{p-1}} = \frac{\left(H_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'}{pz^{p-1}} * z^p g(z).$$

Since

$$Re\,\frac{g(z)}{z^p} > \frac{1}{2},$$

and the function $\frac{1+Az}{1+Bz}$ is convex (univalent) in U, it follows from (8) and Lemma 4 that $(f * g) \in S_{p,m}^{\gamma,k}(\alpha,\beta;A,B)$, which completes the proof of Theorem 5.

Theorem 6. Let $\delta > 0$ and the function $f(z) \in A_p(m)$ satisfying:

$$(1-\delta)\frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{z^p} + \delta\frac{H_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{z^p} \prec \frac{1+Az}{1+Bz}, \tag{42}$$

then

$$Re\left\{\left(\frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{z^p}\right)^{\frac{1}{q}}\right\} > \varepsilon^{\frac{1}{q}}\ (q \in N), \tag{43}$$

where ε in the form (21). The result is the best possible.

Proof Let

$$\varphi(z) = \frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{z^p}, \tag{44}$$

where φ is given by (11). Differentiating (44) and using (16) and (42), we have

$$(1-\delta)\frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{z^p} + \delta\frac{H_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{z^p} = \varphi(z) + \frac{\delta kz\varphi'(z)}{\gamma} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 6 follows by using the technique used in proving Theorem 1. This completes the proof of Theorem 6.

Proceeding on the same lines as in Theorem 6, we can prove the following theorem.

Theorem 7. Let $\delta > 0$ and the function $f(z) \in A_p(m)$ satisfying:

$$(1-\delta)\frac{H_{p,\alpha,\beta+1}^{\gamma,k}f(z)}{z^p} + \delta\frac{H_{p,\alpha,\beta}^{\gamma,k}f(z)}{z^p} \prec \frac{1+Az}{1+Bz},$$

then

$$Re\left\{\left(\frac{H_{p,\alpha,\beta+1}^{\gamma,k}f(z)}{z^p}\right)^{\frac{1}{\sigma}}\right\} > \chi_0^{\frac{1}{\sigma}}\ (\sigma \in N),$$

where

$$\chi_0 = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\beta}{\delta\alpha m}+1;\frac{B}{B-1}\right), & B \neq 0 \\ 1 - \frac{\beta}{\beta+\delta\alpha m}A, & B = 0. \end{cases}$$

The result is the best possible.

Theorem 8. Let $\gamma, v \in C\backslash\{0\}$ and $A, B \in C$ with $A \neq B$ and $|B| \leq 1$. Suppose that

$$\left|\frac{v\gamma(A-B)}{kB} - 1\right| \leq 1 \text{ or } \left|\frac{v\gamma(A-B)}{kB} + 1\right| \leq 1, \text{ if } B \neq 0,$$

$$|v| \leq \frac{k\pi}{\gamma}, \text{ if } B = 0.$$

If $f \in A_p(m)$ with $H_{p,\alpha,\beta}^{\gamma,k} f(z) \neq 0$ for all $z \in U^* = U \backslash \{0\}$, then

$$\frac{H_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{H_{p,\alpha,\beta}^{\gamma,k} f(z)} \prec \frac{1 + Az}{1 + Bz},$$

implies

$$\left(\frac{H_{p,\alpha,\beta}^{\gamma,k} f(z)}{z^p}\right)^v \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1 + Bz)^{\frac{v\gamma}{k}\left(\frac{A-B}{B}\right)}, & B \neq 0 \\ e^{\frac{v\gamma}{k} Az}, & B = 0 \end{cases},$$

is the best dominant (all the powers are the principal ones).

Proof Let us put

$$\varphi(z) = \left(\frac{H_{p,\alpha,\beta}^{\gamma,k} f(z)}{z^p}\right)^v, \tag{45}$$

where the power is the principal one. Then φ is analytic in $U, \varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $z \in U$. Taking the logarithmic derivatives on both sides of (45) and using the identity (6), we have

$$1 + \frac{kz\varphi'(z)}{v\gamma\varphi(z)} = \frac{H_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{H_{p,\alpha,\beta}^{\gamma,k} f(z)} \prec \frac{1 + Az}{1 + Bz}.$$

Now the assertions of Theorem 6 follow by using Lemma 5 with $\lambda = \frac{v\gamma}{k}$. This completes the proof of Theorem 8.

Putting $B = -1$ and $A = 1 - 2\rho, 0 \leq \rho < 1,$ in Theorem 8, we obtain the following result.

Corollary 4. Assume that $\gamma, v \in C \backslash \{0\}$ satisfies either

$$\left|\frac{2v\gamma(1-\rho)}{k} - 1\right| \leq 1 \text{ or } \left|\frac{2v\gamma(1-\rho)}{k} + 1\right| \leq 1.$$

If $f \in A_p(m)$ with $H_{p,\alpha,\beta}^{\gamma,k} f(z) \neq 0$ for all $z \in U^*$, then

$$Re\left(\frac{H_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{H_{p,\alpha,\beta}^{\gamma,k} f(z)}\right) > \rho,$$

implies

$$\left(\frac{H_{p,\alpha,\beta}^{\gamma,k} f(z)}{z^p}\right)^v \prec q_2(z) = (1-z)^{-2v\frac{\gamma}{k}(1-\rho)},$$

and q_2 is the best dominant (the power is the principal one).

4. Remark.

For different value of γ, k, α, β and p in the above results, we obtain results corresponding to the functions given in the introduction.

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