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DIRECT AND CONVERSE THEOREMS OF THE THEORY OF APPROXIMATION IN MORREY SPACES

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Abstract. In this work the approximation properties of the derivatives of the functions by trigonometric polynomials in Morrey spaces are investigated. Some direct and inverse theorems of approximation theory in Morrey spaces are proved.

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1. Introduction

Let T denote the interval $[0,2\pi]$. Let $L^p(T)$, $1\le p<\infty$ be the Lebesgue space of all measurable 2π - periodic functions defined on T such that

$$\|f\|_p := \left(\int_T |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty .$$

The *Morrey spaces* $L_0^{p,\lambda}(T)$ for a given $0\le\lambda\le 1$ and $p\ge 1$, we define as the set of functions $f\in L_{loc}^p(T)$ such that

$$\|f\|_{L_0^{p,\lambda}(T)} := \left\{\sup_I \frac{1}{|I|^{1-\lambda}}\int_I |f(t)|^p\,dt\right\}^{\frac{1}{p}} < \infty$$

where the supremum is taken over all intervals $I\subset[0,2\pi]$. Note that $L_0^{p,\lambda}(T)$ becomes a Banach spaces, $\lambda=1$ coincides with $L^p(T)$ and for $\lambda=0$ with $L^\infty(T)$. If , $0\le\lambda_1\le\lambda_2\le 1$, then $L_0^{p,\lambda_2}(T)\subset L_0^{p,\lambda_1}(T)$. Also, if $f\in L_0^p(T)$, then $f\in L^p(T)$ and hence $f\in L^p(T)$. The Morrey spaces, were introduced by C. B. Morrey in 1938. The properties of the these spaces have been investigated intensively by several authors and together with weighted Lebesgue spaces $L^p_\omega(T)$ play an important role in the theory of partial equations, in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in [14-17], [30], [34], [36], [38], [39] and [46].

In this study we will use the following notations: $\mathbb{N}^+ := \{1, 2, 3, ...\}$, $\mathbb{N}_0 := \mathbb{N}^+ \cup \{0\}$. Also, we shall use $c, c_1, c_2, ...$ to denote constants depending only on parameters that are not important for the questions of our interest

Denote by $C^\infty(T)$ the set of all functions that are realized as the restriction to T of elements in $C^\infty(\square)$. Also we define $L^{p,\lambda}(T)$ to be closure of $C^\infty(T)$ in $L_0^{p,\lambda}(T)$.

Note that in this study we investigate the direct and inverse problems of approximation theory in Morrey space $L^{p,\lambda}(T)$, the closure of the set of trigonometric polynomials in $L_0^{p,\lambda}(T)$ with $1 < p < \infty$.

The function

$$\omega^\alpha_{p,\lambda}(f,\delta) := \sup_{|h|\le\delta} \left\| \Delta^\alpha_h(f,\cdot) \right\|_{L^{p,\lambda}(T)}, \ \alpha \in \square^+$$

is called α -th modulus of smoothness $f \in L^{p,\lambda}(T)$, $0 \le \lambda \le 1$ and $p \ge 1$, where

$$\Delta^\alpha_h(f,\cdot) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} f(x+kh), \ \alpha \in \square^+$$

The modulus of smoothness $\omega^\alpha_{L^{p,\lambda}(T)}(f,\delta)$ have the following properties [24]:

1) $\omega^\alpha_{p,\lambda}(f,\delta)$ is an increasing function,
2) $\lim\limits_{\delta\to 0} \omega^\alpha_{p,\lambda}(f,\delta) = 0$ for every $f \in L^{p,\lambda}(T)$, $0 \le \lambda \le 1$ and $p \ge 1$,
3) $\omega^\alpha_{p,\lambda}(f+g,\delta) \le \omega^\alpha_{p,\lambda}(f,\delta) + \omega^\alpha_{p,\lambda}(f,\delta)$ for $f, g \in L^{p,\lambda}(T)$,
4) $\omega^\alpha_{p,\lambda}(f,n\delta) \le n^\alpha \omega^\alpha_{p,\lambda}(f,\delta)$, $n \in \mathbb{N}_0$,
5) $\omega^\alpha_{p,\lambda}(f,s\delta) \le (s+1)^\alpha \omega^\alpha_{p,\lambda}(f,\delta)$, $s > 0$,
6) $\omega^\alpha_{p,\lambda}(f,\delta) \le \left[(n+1)\delta + 1\right]^\alpha \omega^\alpha_{p,\lambda}\left(f, \frac{1}{n+1}\right)$, $n \in \mathbb{N}_0$.

For $f\in L^{p,\lambda}\left(T\right),$ we say that the function f has derivative f' in the sense of $L^{p,\lambda}\left(T\right)$ if

$$\lim_{h\to 0^{+}}\left\|\frac{\Delta_{h}^{\alpha}\left(f\right)}{h^{\alpha}}-f'\right\|_{L^{p,\lambda}(T)}=0. \tag{1.1}$$

Obviously $f'\in L^{p,\lambda}\left(T\right)$.

Let $S_{n}\left(f\right),\ \left(n=1,2,...\right)$ be the n-th partial sum of the Fourier series of $L^{1}\left(T\right)$, i. e.

$$S_{n}\left(x,f\right)=\frac{a_{0}}{2}+\sum_{k=1}^{n}\left(a_{k}\cos kx+b_{k}\sin kx\right),$$

where a_{k},b_{k} are Fourier coefficients of f.

We denote by $E_{n}\left(f\right)_{L^{p,\lambda}(T)},\ \left(n=0,1,2,...\right)$ the best approximation of $f\in L^{p,\lambda}\left(T\right)$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_{n}\left(f\right)_{L^{p,\lambda}(T)}:=\inf\left\{\left\|f-T_{n}\right\|_{L^{p,\lambda}(T)}:T_{n}\in\Pi_{n}\right\},$$

where Π_{n} denotes the class of trigonometric polynomials of degree at most n.

The problems of approximation of approximation theory in the weighted and non-weighted Morrey spaces have been investigated in [4-6], [8-9], [18], [23], [24], [29], [33] and [37]. In this study a direct theorem of the Jackson type is established in the Morrey spaces $L^{p,\lambda}\left(T\right),\ 0\leq\lambda\leq 1$ and $1<p<\infty$. We also prove a theorem on the relationship the best approximation of the functions with approximation and differential properties of its derivatives in the Morrey spaces $L^{p,\lambda}\left(T\right),\ 0\leq\lambda\leq 1$ and $1<p<\infty$. By using the obtained results the estimation is established for α- th modulus of smoothness of derivative of the functions. This result is the improvement of the result obtained in [29]. Similar problems in different spaces have been investigated by several authors (see, for example, [1-3],[7], [10], [11] [12], [13], [19-22], [25-29], [31], [32], [40], [41],[42], [44], [45], and [47].

Our main results are the following.

Theorem1.1. *If a function* $f\in L^{p,\lambda}\left(T\right),\ 0<\lambda\leq 1$ *and* $1<p<\infty$ *has the derivative of order* $r\in\mathbb{N}_{0}$ *which satisfies the condition* $f^{(r)}\in L^{p,\lambda}\left(T\right),\ 0<\lambda\leq 1$ *and* $1<p<\infty$*, then for every* $\alpha\in\mathbb{R}^{+}$ *the estimate*

$$E_n(f)_{L^{p,\lambda}(T)} \le c_1 n^{-r}\ \omega^{\alpha}_{p,\lambda}\left(f^{(r)},\frac{1}{n}\right),\ n=1,2,... \qquad (1.2)$$

holds with a constant $c_1>0$ *independent of* n.

If $\alpha=1$ this result in the Lebesgue spaces $L^p(T)$, $p=\infty$, $1\le p\le\infty$ was indicated in [3, see, Chapter V], and [26]. In case $r\in\mathbb{N}_0$, $\alpha\in\mathbb{R}^+$ this result in the Lebesgue spaces $L^p(T)$, $p=\infty$ was proved in [41]. If $r=0$ this Theorem in the Morrey space $L^{p,\lambda}(T)$, $0<\lambda\le 1$, and $1<p<\infty$ was proved in [24].

Theorem 1.2. *Let* $E_n(f)_{L^{p,\lambda}(T)} := \|f-T_n\|_{L^{p,\lambda}(T)}$. *If* $f\in L^{p,\lambda}(T)$, $0<\lambda\le 1$ *and* $1<p<\infty$ *and if the condition*

$$\sum_{s=1}^{\infty} s^{\beta r-1}E_n(f)_{L^{p,\lambda}(T)}<\infty$$

is satisfied for some natural number r *and* $\beta=\min(p,2)$, *then* f *has the* r*-th derivative* $f^{(r)}$ *in the sense of (1.1) and the estimate*

$$\left\|f^{(r)}-T_n^{(r)}\right\|_{L^{p,\lambda}(T)}$$
$$\le c_2\left\{n^r E_n(f)_{L^{p,\lambda}(T)}+\left(\sum_{\mu=n+1}^{\infty}\mu^{\beta r-1}E_n^{\beta}(f)_{L^{p,\lambda}(T)}\right)^{1/\beta}\right\},\ n\ge 1,$$

holds with a constant $c_2>0$ *independent of* n.

Corollary 1.1. *Let* $f\in L^{p,\lambda}(T)$, $0<\lambda\le 1$ *and* $1<p<\infty$. *If the condition*

$$\sum_{s=1}^{\infty} s^{\beta r-1}E_n(f)_{L^{p,\lambda}(T)}<\infty$$

is satisfied for some natural number r *and* $\beta=\min(p,2)$, *then* f *has the* r*-th derivative* $f^{(r)}$ *in the sense of (1.1) and the estimate*

$$E_n\left(f^{(r)}\right)_{L^{p,\lambda}(T)}$$
$$\le c_3\left\{n^r E_n(f)_{L^{p,\lambda}(T)}+\left(\sum_{\mu=n+1}^{\infty}\mu^{\beta r-1}E_n^{\beta}(f)_{L^{p,\lambda}(T)}\right)^{1/\beta}\right\},\ n\ge 1 \qquad (1.3)$$

holds with a constant $c_3 > 0$ *independent of* n.

Note that in the Lebesgue space $L^p(T)$, $1 \le p \le \infty$ inequality (1.3) for $r \in \square^+$ was proved without β in [41]. If $r \in \square^+$, then an inequality of type (1.3) in the weighted Lebesgue spaces was proved in [31]. Also, in the particular case for the Lebesgue spaces inequality (1.3) was obtained in [40, (90)]. In case $r \in (0,\infty)$ inequality (1.3) in weighted Lorentz spaces was proved in [47].

Theorem 1.3. *Let* $f \in L^{p,\lambda}(T)$, $0 < \lambda \le 1$ *and* $1 < p < \infty$ *and* $\alpha \in \square^+$. *Assume that the condition*

$$\sum_{s=1}^{\infty} s^{\beta r-1} E_n(f)_{L^{p,\lambda}(T)} < \infty$$

is satisfied for some natural number r *and* $\beta = \min(p,2)$, *then* f *has the* r*-th derivative* $f^{(r)}$ *in the sense of (1.1) and the estimate*

$$\omega_{p,\lambda}^{\alpha}\left(f^{(r)},\frac{1}{n}\right) \le c_4 \frac{1}{n^{\alpha}}\left(\sum_{s=0}^{n}(s+1)^{\beta(\alpha+r)-1} E_n^{\beta}(f)_{L^{p,\lambda}(T)}\right)^{1/\beta} + c_4\left(\sum_{s=n+1}^{\infty} s^{\beta r-1} E_n^{\beta}(f)_{L^{p,\lambda}(T)}\right)^{1/\beta} \tag{1.4}$$

holds with a constant $c_4 > 0$ *independent only on* n.

Note that in cases of $\alpha, r \in \square^+$ and $\alpha, r \in \square^+$ inequality (1.4) in the Lebesgue spaces $L^p(T)$, $1 \le p \le \infty$, was proved without β in [45] and in [42 respectively. (Also, can be see in [41]). Also, in the particular case for the classical Lebesgue spaces inequality (1.4) was proved in [40, (90)]. In case $\alpha, r \in \square^+$ estimate (1.4) in the Morrey spaces $L^{p,\lambda}(T)$, $0 < \lambda \le 1$, and $1 < p < \infty$, was proved without β in [29]. This result is the improvement of the result obtained in study [29].

2. Proofs of main results.

We need the following results.

Theorem 2.1. [24]. *Let* $f \in L^{p,\lambda}(T)$, $0 < \lambda \le 1$ *and* $1 < p < \infty$. *Then for* $\alpha \in \square^+$ the estimate

$$E_n(f)_{L^{p,\lambda}(T)} \le c_5\, \omega_{p,\lambda}^{\alpha}\left(f,\frac{1}{n}\right),\ n = 1,2,...$$

holds with a constant $c_5 > 0$ *independent of* n.

Lemma 2.1. [24]. If $T_n \in \Pi_n$, $n \ge 1$ and $r \in \square^{+}$, then exists a constant $c_6 > 0$ depending only on r, p and λ such that

$$\left\| T_n^{(r)} \right\|_{L^{p,\lambda}(T)} \le c_6 n^r \left\| T_n \right\|_{L^{p,\lambda}(T)}. \tag{2.1}$$

Using Lemma 2.1 about Bernstein inequality related to the trigonometric polynomial T_n of degree $\le n$ in the Morrey spaces $L^{p,\lambda}(T)$, $0 < \lambda \le 1$ and $1 < p < \infty$ and the above emphasized properties of modulus of smoothness $\omega_{p,\lambda}^{\alpha}(f, \delta)$ into account and the proof scheme developed in [44] (see also, [12, p. 210] we can prove the following theorem.

Theorem 2.2. *Let* $f \in L^{p,\lambda}(T)$, $0 < \lambda \le 1$ *and* $1 < p < \infty$. *Then for a given* $\alpha \in \square^{+}$ the estimate

$$\omega_{p,\lambda}^{\alpha}\left(f, \frac{1}{n}\right) \le \frac{c_7}{n^{\alpha}} \left\{ \sum_{s=0}^{\infty} (s+1)^{\beta\alpha - 1} E_s^{\beta}(f)_{L^{p,\lambda}(T)} + \right\}^{1/\beta}$$

holds, where $\beta = \min\{p, 2\}$ *and the constant* $c_7 > 0$ *independent of* n.

Also, using the proof scheme developed in [35, Theorem 1] we can prove following theorem related to the Littlewood- Paley inequality in the Morrey spaces $L^{p,\lambda}(T)$, $0 < \lambda \le 1$ and $1 < p < \infty$.

Theorem 2.3. *Let* $f \in L^{p,\lambda}(T)$, $0 < \lambda \le 1$ *and* $1 < p < \infty$.*Then there exist constants* c_8 *and* c_9 *depending only on* p *and* λ *such that*

$$c_8 \left\| \left(\sum_{\mu=\nu}^{\infty} B_{\mu}^2(f,x) \right)^{\frac{1}{2}} \right\|_{L^{p,\lambda}(T)} \le \left\| \sum_{\mu=2^{\nu-1}}^{\infty} A_{\mu}(f,x) \right\|_{L^{p,\lambda}(T)} \le c_9 \left\| \left(\sum_{\mu=\nu}^{\infty} B_{\mu}^2(f,x) \right)^{\frac{1}{2}} \right\|_{L^{p,\lambda}(T)}$$

for every $\nu \in \square^{+}$, *where*

$$B_{\mu}(f,x) := \sum_{\nu=2^{\mu-1}}^{2^{\eta}-1} A_{\nu}(f,x), \; \mu = 1, 2, \ldots, \; A_{\nu}(f,x) := a_{\nu} \cos \nu x + b_{\nu} \sin \nu x.$$

Note that, Theorem 2.2 and 2.3 have been proved in the thesis entitled “ N. P. Tozman, Some problems of approximation theory in Morrey spaces, PhD thesis, Balıkesir University, Graduate School of Natural and Applied Sciences, Balıkesir, Turkey, (2009), (in Turkish)”.

Proof of Theorem 1.1. The following formula holds:

$$\Delta_h^{\alpha+r} f(x) = \int_0^h dt_1 \ldots \int_0^h \Delta_h^{\alpha+r} f^{(r)}(x+t_1+t_2+\ldots+t_r)\,dt_r. \tag{2.2}$$

Using the definition of α -th modulus of smoothness of $\omega_{p,\lambda}^{\alpha}(f,\delta)$, from (2.2) we obtain inequality

$$\omega_{p,\lambda}^{\alpha+r}(f,\delta) \le \delta^r \omega_{p,\lambda}^{\alpha}\left(f^{(r)},\delta\right),\ \delta > 0.$$

Then from the last relation and Theorem 2.1 we obtain inequality (1.2) of Theorem 1.1.

Proof of Theorem 1.2. There exist a sequence of trigonometric polynomials $\{T_n\}_{n=1}^{\infty}$ such that

$$\| f - T_n \|_{L^{p,\lambda}(T)} = E_n(f)_{L^{p,\lambda}(T)}.$$

From the conditions of theorem the following expressions holds:

$$\left\| T_{2^i} - T_{2^{i-1}} \right\|_{L^{p,\lambda}(T)} \le c_{10} E_{2^{i-1}}(f)_{L^{p,\lambda}(T)},$$

$$f = T_1 + \sum_{i=1}^{\infty}\left(T_{2^i} - T_{2^{i-1}}\right) = \sum_{i=0}^{\infty} V_{2^i}.$$

Note that where the convergence is understood in the sense of $L^{p,\lambda}(T)$.

Now, we show that for $j = 1,\ldots,r$ there exist the functions $\psi_j(x) \in L^{p,\lambda}(T)$ such that

$$\psi_j(x) = \sum_{i=0}^{\infty} V_{2^i}^{(j)}(x)$$

and

$$\psi_j(x) = f^{(j)}(x).$$

Using (2.1) for $j = 1$ we obtain

$$\left\|\frac{f(\cdot+h)-f(\cdot)}{h}-\psi_1(\cdot)\right\|_{L^{p,\lambda}(T)}$$

$$\leq\left\|\sum_{i=0}^{\infty}\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-\sum_{i=0}^{\infty}V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(T)}$$

$$\leq\sum_{i=0}^{n_0}\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(T)}$$

$$+\sum_{i=n_0}^{\infty}\left(\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(T)}+\left\|V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(T)}\right)$$

$$\leq\sum_{i=0}^{n_0}\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(T)}+c_{11}\sum_{i=n_0+1}^{\infty}2^{i\beta}\left\|V_{2^i}\right\|_{L^{p,\lambda}(T)}. \quad (2.3)$$

From the inequality (2.3) for $h\to 0$ and $n\geq n_0$ we have

$f'(x)=\psi_1(x)$.

For $j=2,...,n$, to prove theorem we use the method of induction.

According to [22] the inequality

$$\left\|f^{(r)}-S_n\left(f^{(r)}\right)\right\|_{L^{p,\lambda}(T)}\leq c_{12}E_n\left(f^{(r)}\right)_{L^{p,\lambda}(T)} \quad (2.4)$$

holds.

Let us choose m such that $2^m\leq n<2^{m+1}$. The inequality

$$\left\|f^{(r)}-T_n^{(r)}\right\|_{L^{p,\lambda}(T)}\leq\left\|T_n^{(r)}-T_{2^{m+2}}^{(r)}\right\|_{L^{p,\lambda}(T)}+\sum_{i=m+2}^{\infty}\left\|T_{2^{i+1}}^{(r)}-T_{2^i}^{(r)}\right\|_{L^{p,\lambda}(T)} \quad (2.5)$$

holds.

Using (2.1) we have

$$\left\|T_n^{(r)}-T_{2^{m+2}}^{(r)}\right\|_{L^{p,\lambda}(T)}\leq c_{13}2^{(m+2)r}E_{2^{m+2}}(f)_{L^{p,\lambda}(T)}$$

$$\leq c_{14}n^rE_n(f)_{L^{p,\lambda}(T)}. \quad (2.6)$$

We use the notation $\lambda_{j,\mu}:=\sum_{\nu=1}^{j}B_{\nu,\mu}$, where $B_{\nu,\mu}=a_\nu\cos\left(\nu+\mu\frac{\pi}{2}\right)x+b_\nu\sin\left(\nu+\mu\frac{\pi}{2}\right)x$. Using Abel's transformation we get

$$\sum_{\nu=2^i+1}^{2^{i+1}} \nu^\mu B_{\nu,\mu}(x) = \sum_{\nu=2^i+1}^{2^{i+1}} \left[\nu^\mu - (\nu+1)^\mu\right]\left(\lambda_{\nu,\mu}(x) - \lambda_{2^i,\mu}(x)\right)$$
$$+ 2^{i+1}\left(\lambda_{2^{i+1},\mu}(x) - \lambda_{2^i,\mu}(x)\right).$$

The last relation and (2.4) imply that

$$\left\|\sum_{\nu=2^i+1}^{2^{i+1}} \nu^\mu B_{\nu,\mu}(x)\right\|_{L^{p,\lambda}(T)} \le c_{15} \sum_{\nu=2^i+1}^{2^{i+1}} \left[(\nu+1)^\mu - \nu^\mu\right] E_{2^i}(f)_{L^{p,\lambda}(T)}$$
$$+ c_{16} 2^{i\mu} E_{2^i}(f)_{L^{p,\lambda}(T)} \le c_{17} 2^{i\mu} E_{2^i}(f)_{L^{p,\lambda}(T)}. \qquad (2.7)$$

According to Theorem 2.3 and inequality (2.7) we conclude that

$$\sum_{i=m+2}^{\infty} \left\|T_{2^{i+1}}^{(r)} - T_{2^i}^{(r)}\right\|_{L^{p,\lambda}(T)} \le c_{18} \left(\sum_{i=m+2}^{\infty} \left\| \sum_{\mu=2^i+1}^{2^{i+1}} \mu^r B_{\mu,r}(x) \right\|_{L^{p,\lambda}(T)}^{\beta} \right)$$
$$\le c_{19} \left(\sum_{i=m+2}^{\infty} 2^{ir\beta} E_{2^i-1}^{\beta}(f)_{L^{p,\lambda}(T)} \right)^{\frac{1}{\beta}}$$
$$\le c_{20} \left(\sum_{i=n+1}^{\infty} i^{r\beta-1} E_i^{\beta}(f)_{L^{p,\lambda}(T)} \right)^{\frac{1}{\beta}}. \qquad (2.8)$$

From (2.5), (2.6) and (2.8), we conclude the required result:

$$\left\| f^{(r)} - T_n^{(r)} \right\|_{L^{p,\lambda}(T)} \le c_{21} \left\{ n^r E_n(f)_{L^{p,\lambda}(T)} + \left(\sum_{\mu=m+1}^{\infty} \mu^{r\beta} E_\mu^{\beta}(f)_{L^{p,\lambda}(T)} \right)^{\frac{1}{\beta}} \right\}.$$

This completes the proof of Theorem 1.2.

Corollary 1.1 follows immediately from Theorem 1.2.

Theorem 1.3 follows immediately from Theorem 2.2 and 1.2.

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