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SOME NEW SEMI-NORMED SEQUENCE SPACES OF NON-ABSOLUTE TYPE AND MATRIX TRANSFORMATIONS

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Abstract. The purpose of the present study is to introduce the sequence spaces. We investigate some topological properties of these spaces and also establish some inclusion relations between them. Furthermore, we compute α- and β-duals of these spaces and characterize the classes of infinite matrices.

Keywords: Semi-normed sequence spaces, matrix domains, α- and β-duals, matrix transformations.

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1. Introduction

Let ω denotes the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. We write l_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs and l_1, we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

Let X, Y be two sequence spaces and $\mathrm{A} = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N} = \{1, 2, \cdots\}$. We say that A defines a matrix mapping from X into Y, and we denote it by $A: X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$ exists and is in Y, where $(Ax)_n = \sum_{n=1}^{\infty} a_{nk}x_k$ for $n = 1, 2, \cdots$. By (X, Y), we denote the class of all infinite matrices A such that $A: X \to Y$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_n) \in \omega : \ Ax \in X\}, \tag{1}$$

which is a sequence space. The new sequence space X_A generated by the limitation matrix A from a sequence space X can be the expansion or the contraction and or the overlap of the original space X. A matrix $\mathrm{A} = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$.

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of triangle matrices in classical spaces l_p, l_∞, c and c_0. For instance, some matrix domains of the difference operator were studied in [4, 8, 9, 13], of the Riesz matrices in [1, 3], of the Euler matrices in [2, 6, 12], of the Cesáro matrices in [5, 14, 15], and of the Nörlund matrices in [16, 17]. In these studies the matrix domains are obtained by triangle matrices, hence these spaces

are normed sequence spaces. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [7]. The matrix domains given in this paper specify by a certain non-triangle matrix, so we should not expect that related spaces are normed sequence spaces.
In this study, the normed sequence space X is extended to semi-normed space $X(E)$, where $X \in \{l_\infty, c, c_0\}$. We consider some topological properties of the spaces $l_\infty(E)$, $c(E)$ and $c_0(E)$, and derive some inclusion relations concerning with them. Furthermore, we determine the $\alpha-$ and β-duals for these spaces. Finally, we obtain the necessary and sufficient conditions on an infinite matrix belonging to the classes $(X(E);\ l_\infty)$, $(X(E), c)$ and $(X(E), c_0)$, where $X \in \{l_\infty, c, c_0\}$.

The results are generalizations of some results of Malkowsky and Rakocevic [11]. In a similar way, the second Author has introduced the sequence spaces $l_p(E)$, where $1 \le p < \infty$, [10].

2. The sequence spaces $X(E)$ for $X \in \{l_\infty, c, c_0\}$

Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$maxE_n < minE_{n+1}, \tag{2}$$

$n = 1, 2, \cdots$. We define the sequence spaces $X(E)$ for $X \in \{l_\infty, c, c_0\}$ by

$$X(E) = \left\{ x = (x_k)_{k=1}^{\infty} \in \omega \ :\ \left(\sum_{i \in E_k} x_i\right)_{k=1}^{\infty} \in X \right\}.$$

$X(E)$ is a semi-normed space with the semi-norm $\|.\|_E$, which is defined by the following way:

$$\|x\|_E = \sup_k \left|\sum\nolimits_{i \in E_k} x_i\right|. \tag{3}$$

It should be noted that the function $\|.\|_E$ cannot be the norm, since if $x = (1, -1, 0, 0, \cdots)$ and $E_n = \{2n-1,\ 2n\}$ for all n, then $\|x\|_E = 0$ while $x \neq 0$. It is also significant that in the special case $E_n = \{n\}$ for $= 1, 2, \cdots$, we have $X(E) = X$ and $\|x\|_E = \|x\|_\infty$, where $x \in X$ and $\|x\|_\infty = sup_n|x_n|$ is the usual norm of the spaces l_∞, c and c_0.

Suppose $E = (E_n)$ is a sequence of finite subsets of the positive integers that satisfies the condition (2). If the infinite matrix $\mathrm{A} = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} 1 & if\ k \in E_n \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

with the notation of (1), we can redefine the spaces $l_\infty(E), c(E)$ and $c_0(E)$ as follows:

$$l_\infty(E) = (l_\infty)_A \quad and \quad c(E) = (c)_A \quad and \quad c_0(E) = (c_0)_A.$$

Now, we may begin with the following theorem which is essential in the study.

Theorem 1. The sequence spaces $X(E)$ for $X \in \{l_\infty, c, c_0\}$ are complete semi-normed vector spaces with respect to the semi-norm defined by (3).

Proof. This is a routine verification and so we omit the detail.
It can easily be checked that the absolute property does not hold on the space $X(E)$, that is $\|x\|_E \neq \||x|\|_E$ for at least one sequence in this space which says that $X(E)$ is the sequence space of non-absolute type, where $|x| = (|x_k|)$.
Throughout this article, we denote the cardinal number of the set E_k by $|E_k|$.
Theorem 2. Let $M = \left\{ x = (x_n)_{n=1}^{\infty} \ : \ \sum_{j \in E_n} x_j = 0, \ \forall n \right\}$. The quotient spaces $l_\infty(E)/M$, $c(E)/M$ and $c_0(E)/M$ are linearly isomorphic to the spaces l_∞, c and c_0, respectively.
Proof. Let $X \in \{l_\infty, c, c_0\}$. Consider the map $T : X(E) \to X$ defined by

$$Tx = \left(\sum_{j \in E_n} x_j \right)_{n=1}^{\infty}$$

for all $x \in X(E)$. The linearity of T is trivial. Let $y \in X$ and $\alpha_n = |E_n|$ for all n. We define the sequence $x = (x_k)$ by $x_k = y_n/\alpha_n$ for all $k \in En$. It is clear that $x \in X(E)$ and $Tx = y$. Thus the map T is surjective. By applying the first isomorphism theorem we have $X(E) = M \,/\, X$, because $kerT = M$.

Note that the mapping defined in Theorem 2, T is not injective. While the

$$\|Tx\|_\infty = \|x\|_E,$$

for all $x \in X(E)$.
Definition 1. Let $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2), and $s = (s_n)$ be a strictly increasing sequence of the positive integers. The generated partition $H = (H_n)$ is defined by E and s, as follows

$$H_n = \bigcup_{j=s_{n-1}+1}^{s_n} E_j,$$

for $n = 1,2,\cdots$.

Here and in the sequel, we shall use the convention that any term with a zero subscript is equal to zero. Note that any arbitrary partition $H = (H_n)$ that satisfies the condition (2) generate by the partition $E = (E_n)$ and the sequence $s = (s_n)$, where $E_n = \{n\}$ and $s_n = \, max\, H_n$ for all n. It is also important to know $s_n - s_{n-1} = |H_n|$.

In the following, the inclusion relation between the spaces $X(E)$ and $X(H)$ is examined. Obviously if $s_n - s_{n-1} > 1$ only for a finite number of n, then

$$X(E) = X(H).$$

Especially if $|H_n| > 1$ only for a finite number of n, then $X = X(H)$, where $X \in \{l_\infty, c, c_0\}$.
Theorem 3. Let E, s and H be as in Definition 1. Then, the following statements hold:
(i) If

$$\sup_n (s_n - s_{n-1}) < \infty,$$

we have $l_\infty(E) \subset l_\infty(H)$, and $c_0(E) \subset c_0(H)$.
(ii) If there is a positive integer ξ such that $s_n = \xi n$ for all n, then $c(E) \subset c(H)$.
(iii) Moreover if $s_n - s_{n-1} > 1$ for an infinite number of n, then the inclusion relations in parts (i) and (ii) are strict.

Proof. (i) Suppose that $\zeta = sup_n(s_n - s_{n-1})$ and $x = (x_k) \in l_\infty(E)$, we have

$$\sup_n \left| \sum_{j\in H_n} x_j \right| = \sup_n \left| \sum_{k=s_{n-1}+1}^{s_n} \sum_{j\in E_k} x_j \right| \leq \zeta \sup_k \left| \sum_{j\in E_k} x_j \right| < \infty.$$

This shows $x = (x_k) \in l_\infty(H)$, so $l_\infty(E) \subset l_\infty(H)$. Since

$$\sum_{j\in H_n} x_j = \sum_{k=s_{n-1}+1}^{s_n} \sum_{j\in E_k} x_j, \tag{5}$$

for all n, we can conclude $c_0(E) \subset c_0(H)$.
(ii) The inclusion $c(E) \subset c(H)$ holds, since $s_n - s_{n-1} = \xi$ for all n.
(iii) By assumption $s_n - s_{n-1} > 1$ for an infinite number of n, one can choose a subsequence (n_j) in $\mathbb{N}$ with $s_{n_j} - s_{n_j-1} > 1$ for $j = 1, 2, \cdots$. We define the sequence $x = (x_k)$ such that

$$\sum_{j\in E_k} x_j = \begin{cases} j & if\ k = s_{n_j-1} + 1 \\ j & if\ k = s_{n_j-1} + 2 \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for $k = 1, 2, \cdots$. It is obvious that $\sum_{i\in H_k} x_i = 0$, so $x \in X(H)$ while the $x \notin X(E)$, for $X \in \{l_\infty, c, c_0\}$. Hence the inclusions in parts (i) and (ii) are strict.

Corollary 1. Let $H = (H_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). Then, the following statements hold:
(i) If $\sup_n |H_n| < \infty$, we have $l_\infty \subset l_\infty(H)$, and $c_0 \subset c_0(H)$.
(ii) If there is a positive integer ξ such that $|H_n| = \xi n$ for all n, then $c \subset c(H)$.
(iii) Moreover if $|H_n| > 1$ for an infinite number of n, then the inclusion relations in parts (i) and (ii) are strict.

Proof. If $E_n = \{n\}$ and $s_n = max\, H_n$ for all n, then the partition $H = (H_n)$ is generated by $E = (E_n)$ and $s = (s_n)$. The desired result follows from Theorem 3.

Corollary 2. Let M and N be two positive integers. If we put $E_i = \{Mi - M + 1, Mi - M + 2, \cdots, Mi\}$ and $H_i = \{MNi - MN + 1, MNi - MN + 2, \cdots, MNi\}$ for all i, then $X(E) \subset X(H)$, where $X \in \{l_\infty, c, c_0\}$. Moreover if $N > 1$, then these inclusions are strict.

Proof. If $s_i = Ni$ for all i, then the partition $H = (H_n)$ is generated by E and s. The desired result follows from Theorem 3.

3. The $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-duals of the sequence space $\boldsymbol{X(E)}$

In this section, we compute the α- and β-duals for the sequence spaces $l_\infty(E), c(E)$ and $c_0(E)$. For the sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y) = \{a = (a_k) \in \omega : \ (a_k x_k)_{k=1}^{\infty} \in Y \ \ \forall x = (x_k) \in X\}$$

is called the multiplier space of X and Y. With the above notation, the α- and β-duals of a sequence space X, which are respectively denoted by X^α and X^β, are defined by

$$X^\alpha = M(X, l_\infty), \qquad\qquad X^\beta = M(X, cs).$$

Lemma 1 [11]. Let $X, Y, Z \subset \omega$. We have
(i) $X \subset Z$ implies $M(Z, Y) \subset M(X, Y)$.
(ii) $Y \subset Z$ implies $M(X,Y) \subset M(X,Z)$. In particular $X^\alpha \subset X^\beta$.

Theorem 4. Define the set d as follows:

$$d = \left\{ a = (a_k) \in \omega \ : \ \sum_{k=1}^{\infty} \left(\sup_{i \in E_k} |a_i| \right) < \infty \right\},$$

then

$$(c_0(E))^\beta = (c(E))^\beta = (l_\infty(E))^\beta = d.$$

Proof. Obviously $(l_\infty(E))^\beta \subset (c(E))^\beta \subset (c_0(E))^\beta$ by Part (i) of Lemma 1. So it is sufficient to verify the inclusions $d \subset (l_\infty(E))^\beta$ and $(c_0(E))^\beta \subset d$.
Let $a \in d$ be given. Since $E = (E_n)$ is a partition of the positive integers, we have

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_k x_k \right| &= \left| \sum_{k=1}^{\infty} \sum_{i \in E_k} a_i x_i \right| \\ &\le \sum_{k=1}^{\infty} \left(\sup_{i \in E_k} |a_i| \right) \left| \sum_{i \in E_k} x_i \right| \\ &\le \sup_k \left| \sum_{i \in E_k} x_i \right| \sum_{k=1}^{\infty} \left(\sup_{i \in E_k} |a_i| \right) < \infty, \end{aligned} \tag{7}$$

for all $x \in l_\infty(E)$. This shows $ax \in cs$, thus $a \in (l_\infty(E))^\beta$ and hence $d \subset (l_\infty(E))^\beta$.
Now, let $a \in (l_\infty(E))^\beta$ be given. We consider the linear functional $f_n : c_0(E) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{n} \sum_{i \in E_k} a_i x_i \qquad\qquad (x = (x_k) \in c_0(E)),$$

for $n = 1, 2, \cdots$. Similar to (7), we obtain

$$|f_n(x)| \leq \sup_k \left|\sum_{i\in E_k} x_i\right| \sum_{k=1}^{n} \left(\sup_{i\in E_k}|a_i|\right),$$

for every $x \in c_0(E)$. So the linear functional f_n is bounded and $\|f_n\| \leq \sum_{k=1}^{n}\left(\sup_{i\in E_k}|a_i|\right)$. We now prove reverse of the above inequality. Without loss of generality we assume there is an index i such that $1 \leq i \leq maxE_n$ and $a_i \neq 0$, since the case $a_i = 0$ for all $1 \leq i \leq maxE_n$ is trivial. We define the sequence $x = (x_i)$ by $x_i = \operatorname{sgn} a_i$, where $i \in E_k$ is the first index of E_k such that $|a_i| = \sup_{j\in E_k}|a_j|$ for $1 \leq k \leq n$, and put the remaining elements zero. Obviously $x \in c_0(E)$, so

$$\|f_n\| \geq \frac{|f_n(x)|}{\|x\|_E} = \sum_{k=1}^{n}\left(\sup_{j\in E_k}|a_j|\right),$$

and $\|f_n\| = \sum_{k=1}^{n}\left(\sup_{j\in E_k}|a_j|\right)$ for $n = 1, 2, \cdots$. Since $a \in (c_0(E))^\beta$, the map $f_a : c_0(E) \to \mathbb{R}$ defined by

$$f_a(x) = \sum_{k=1}^{\infty}\sum_{i\in E_k} a_i x_i \qquad (x = (x_k) \in c_0(E))$$

is well-defined and linear, and also the sequence (f_n) is pointwise convergent to f_a. By using the Banach-Steinhaus theorem, it can be shown that $\|f_a\| \leq \sup_n\|f_n\| < \infty$, so $\sum_{k=1}^{\infty}\left(\sup_{i\in E_k}|a_i|\right) < \infty$ and $a \in d$. This completes the proof.

It is clear that $d = l_1$ when $\sup_k|E_k| < \infty$, since

$$\sum_{k=1}^{\infty}\left(\sup_{i\in E_k}|a_i|\right) \leq \sum_{k=1}^{\infty}|a_k|,$$

and

$$\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\sum_{i\in E_k}|a_i| = \left(\sup_k|E_k|\right)\sum_{k=1}^{\infty}\left(\sup_{i\in E_k}|a_i|\right).$$

When $\sup_k|E_k| = \infty$, we have $l_1 \subset d$ and the inclusion is strict. Since if $E_1 = \{1,2\}$ and $E_n = \{\min E_{n-1} + n(n+1), \cdots, \max E_n + n(n+1)\}$, (n = 2, 3, ⋯), and if the sequence $x = (x_k)$ is chosen such that $x_k = \frac{1}{n(n+1)}$ for $k \in E_n$. It is obvious that $x \in d$ while $x \notin l_1$.

Corollary 3. Let $\sup_k|E_k| < \infty$. Then, we have

$$(c_0(E))^\beta = (c(E))^\beta = (l_\infty(E))^\beta = l_1.$$

Proof. Since $d = l_1$, by using Theorem 4, we obtain the desired result.

Corollary 4 [11]. We have ${c_0}^{\beta} = c^{\beta} = {l_\infty}^{\beta} = l_1$.
Proof. If $E_k = \{k\}$ for all k, by applying Corollary 3, we obtain the desired result.
Theorem 5. We have $(X(E))^{\alpha} \subset d$, where $X \in \{l_\infty, c, c_0\}$. Moreover if $|E_n| > 1$ for an infinite number of n, then these inclusions are strict.
Proof. One can conclude from Theorem 4 with Part (ii) of Lemma 1 that

$$(X(E))^{\alpha} \subset \left(X(E)\right)^{\beta} = d,$$

where $X \in \{l_\infty, c, c_0\}$. Moreover, if the sequence $a = (a_i)$ is defined by $a_i = 1/2^{n-1}$ whenever $i \in E_n$, then we have

$$\sum_{n=1}^{\infty}\left(\sup_{i\in E_n}|a_i|\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} < \infty,$$

which means that $a \in d$. Because $|E_n| > 1$ for an infinite number of n, we may choose an index subsequence (n_j) of the positive integers with $\left|E_{n_j}\right| > 1$ for $j = 1, 2, \cdots$. Let $\alpha_j = \min E_{n_j}$, we define the sequence $x = (x_i)$ as follows:

$$x_i = \begin{cases} 2^{n_j-1} & if\ i = \alpha_j \\ -2^{n_j-1} & if\ i = \alpha_j + 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \cdots$. Thus $\sum_{i\in E_k} x_i = 0$ for all k, and $x \in X(E)$. But

$$\sum_{k=1}^{\infty}|a_k x_k| = \sum_{j=1}^{\infty}\sum_{i\in E_{n_j}}|a_i x_i| = \sum_{j=1}^{\infty} 2 = \infty,$$

which shows that $a \notin (X(E))^{\alpha}$ for $X \in \{l_\infty, c, c_0\}$. Therefore, these inclusions strictly hold. This step completes the proof.
Corollary 5. Let $\sup_k |E_k| < \infty$. We have $(X(E))^{\alpha} \subset l_1$, where $X \in \{l_\infty, c, c_0\}$. Moreover if $|E_n| > 1$ for an infinite number of n, then these inclusions are strict.
Proof. Since $d = l_1$, by using Theorem 5, we obtain the desired result.

4. Matrix transformations on sequence spaces $X(E)$

In the present section, some classes of infinite matrices related with new sequence spaces are characterized. Let $A = (a_{nk})$ be an infinite matrix of real numbers and X and Y be two sequence spaces. We write $A_n = (a_{nk})_{k=1}^{\infty}$ for the sequence in the n-th row of A. It is clear that $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all n and $Ax \in Y$ for all $x \in X$.
We start with the following lemma which is needed to prove our main results.
Lemma 2. If $a = (a_k) \in d$, then the linear functional $f : c_0(E) \to \mathbb{R}$ defined by

$$f(x)=\sum_{k=1}^{\infty}a_kx_k \qquad (x=(x_k)\in c_0(E)),$$

is bounded and

$$\|f\|=\sum_{n=1}^{\infty}\left(\sup_{i\in E_n}|a_i|\right).$$

Proof. Since $f(x)=\sum_{k=1}^{\infty}\sum_{i\in E_k}a_ix_i$, the proof is obtained by the proof of Theorem4.

Let $A=(a_{nk})$ be an infinite matrix. We consider the conditions

$$\sup_n\left(\sum_{k=1}^{\infty}\sup_{i\in E_k}|a_{ni}|\right)<\infty, \tag{8}$$

$$\lim_{n\to\infty}\sup_{i\in E_k}a_{ni}=0 \quad (k=1,2,\cdots), \tag{9}$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\sup_{i\in E_k}a_{ni}\right)=0, \tag{10}$$

$$\lim_{n\to\infty}\sup_{i\in E_k}a_{ni}=l_k \text{ for some } l_k\in\mathbb{R} \ (k=1,2,\cdots), \tag{11}$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\sup_{i\in E_k}a_{ni}\right)=l \text{ for some } l\in\mathbb{R}. \tag{12}$$

Theorem 6. We have
(i) $(c_0(E),l_\infty)=(c(E),l_\infty)=(l_\infty(E),l_\infty)$ and $A\in(l_\infty(E),l_\infty)$ if and only if the condition (8) holds;
(ii) $A\in(c_0(E),c_0)$ if and only if the conditions (8) and (9) hold;
(iii) $A\in(c(E),c_0)$ if and only if the conditions (8), (9) and (10) hold;
(iv) $A\in(c_0(E),c)$ if and only if the conditions (8) and (11) hold;
(v) $A\in(c(E),c)$ if and only if the conditions (8), (11) and (12) hold.

Proof. (i) First we show that $A\in(l_\infty(E),l_\infty)$ if and only if the condition (8) holds. By the condition (8), we have $A_n\in d$. So due to Theorem 4, $A_n\in\left(l_\infty(E)\right)^\beta$ for $n=1,2,\cdots$. Let $x\in l_\infty(E)$. Similar to the proof of Lemma 2, we deduce that

$$|A_n(x)|\le\left(\sum_{k=1}^{\infty}\left|\sup_{i\in E_k}a_{ni}\right|\right)\|x\|_E,$$

for all n. This implies that $Ax\in l_\infty$ for each $x\in l_\infty(E)$, so $A\in(l_\infty(E),l_\infty)$.

Conversely let $A\in(l_\infty(E),l_\infty)$. We define the sequence $x=(x_i)$ by $x_i=\operatorname{sgn}a_{ni}$, where $i\in E_k$ is the first index of E_k such that $|a_{ni}|=\sup_{j\in E_k}|a_{nj}|$, and put the remaining elements zero. It is clear that $x\in l_\infty(E)$, so $Ax\in l_\infty$ and the condition (8) must hold.

Now we show that $A \in (c_0(E), l_\infty)$ if and only if the condition (8) holds. Like the previous part it's clear that $A \in (c_0(E), l_\infty)$ if the condition (8) holds. Conversely, suppose that $A \in (c_0(E), l_\infty)$. By our hypothesis that $Ax \in l_\infty$ whenever $x \in c_0(E)$, it can be concluded that $\sum_{k=1}^{\infty} \sup_{l \in E_k} |a_{nl}| < \infty$ for all n; Otherwise if $\sum_{k=1}^{\infty} \left(\sup_{l \in E_k} |a_{n'l}| \right) = \infty$, for some positive integer n'. There is a strictly increasing sequence $(m_j)_{j=1}^{\infty}$ of positive integers such that

$$\sum_{k=m_j+1}^{m_{j+1}} \left(\sup_{l \in E_k} |a_{n'l}| \right) > j.$$

We define the sequence $x = (x_i)$ by $x_i = \frac{\operatorname{sgn} a_{n'i}}{j}$, where $i \in E_k$ is the first index of E_k such that $|a_{n'i}| = \sup_{l \in E_k} |a_{n'l}|$, $m_j < k \le m_{j+1}$, and put the remaining elements zero, especially $x_k = 0$ for $0 \le k \le m_1$. It is obvious that $x \in c_0(E)$, and

$$\sum_{k=1}^{\infty} a_{n'k} x_k = \sum_{j=1}^{\infty} \sum_{k=m_j+1}^{m_{j+1}} \frac{\sup_{l \in E_k} |a_{n'l}|}{j} > \sum_{j=1}^{\infty} 1.$$

This means that $\sum_{k=1}^{\infty} a_{n'k} x_k$ is divergent, which contradicts our assumption. Hence $\sum_{k=1}^{\infty} \left(\sup_{l \in E_k} |a_{n'l}| \right) < \infty$ and $A_n \in d$, for all n. Due to Lemma 2, the linear functional $f_n : c_0(E) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k \qquad (x = (x_k) \in c_0(E)),$$

is bounded and $\|f_n\| = \sum_{k=1}^{\infty} \sup_{l \in E_k} |a_{n'l}|$, for all n. By applying the Banach-Steinhaus theorem it follows that $\sup_n \|f_n\| < \infty$, so $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and the condition (8) must hold. The proof is completed according to the mentioned topics and $(l_\infty(E), l_\infty) \subseteq (c(E), l_\infty) \subseteq (c_0(E), l_\infty)$.

(ii) Suppose that $A \in (c_0(E), c_0)$. We define the sequence $e^k = (e_i^k)_{i=1}^{\infty}$ by $e_i^k = 1$, where $i \in E_k$ is the first index of E_k such that $|a_{ni}| = \sup_{l \in E_k} |a_{nl}|$ for $k = 1, 2, \cdots$, and put the remaining elements zero. Obviously $e^k \in c_0(E)$ and $Ae^k \in c_0$, this proves the necessity of the condition (9). The proof of the necessity of the condition (8) is similar to previous part. Conversely, suppose that the conditions (8) and (9) hold. For every $x \in c_0(E)$

$$
\begin{aligned}
|A_n(x)| &= \left|\sum_{k=1}^{\infty}\sum_{i\in E_k} a_{ni}x_i\right| \\
&\leq \sum_{k=1}^{m}\sup_{i\in E_k}|a_{ni}|\left|\sum_{i\in E_k} x_i\right| + \sum_{k=m+1}^{\infty}\sup_{i\in E_k}|a_{ni}|\left|\sum_{i\in E_k} x_i\right| \\
&\leq \|x\|_E\sum_{k=1}^{m}\sup_{i\in E_k}|a_{ni}| + \sup_{k\geq m+1}\left|\sum_{i\in E_k} x_i\right|\sum_{k=m+1}^{\infty}\sup_{i\in E_k}|a_{ni}|.
\end{aligned}
$$

Now take m so large that $\sup\limits_{k\geq m+1}\left|\sum_{i\in E_k} x_i\right| < \epsilon$ and then take n so large that $\sum_{k=1}^{m}\sup\limits_{i\in E_k}|a_{ni}| < \epsilon$ (possible since $\lim\limits_{n\to\infty}\sup\limits_{i\in E_k}|a_{ni}| = 0$). We have $Ax \in c_0$.

(iii) Suppose that $A \in (c(E), c_0)$. Since the sequences $e^k \in c(E)$, we deduce that $Ae^k \in c_0$ for $k = 1, 2, \cdots$. This implies that $(\sup\limits_{i\in E_k}|a_{ni}|)_{k=1}^{\infty} \in c_0$, so the condition (9) holds. Also if the sequence $e = (e_i)$ is defined by $e_i = 1$ when $a_{ni} = \sup\limits_{l\in E_k}|a_{nl}|$ for some k, and we put the remaining elements zero. Then $e \in c(E)$ indicate that $Ae = \left(\sum_{k=1}^{\infty}\sup\limits_{l\in E_k}|a_{nl}|\right)_{n=1}^{\infty} \in c_0$, so the condition (10) is satisfied. The proof of the necessity of the condition (8) is similar to the part (i).

Conversely, suppose that the conditions (8), (9) and (10) hold. For every $x \in c(E)$ there is a finite number as l such that $\lim\limits_{k\to\infty}\sum_{i\in E_k} x_i = 0$. If the sequence $y = (y_i)$ defined by $y_i = \frac{l}{|E_k|}$ for $i \in E_k$, then

$$
A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k = \sum_{k=1}^{\infty} a_{nk}(x_k - y_k) + \sum_{k=1}^{\infty} a_{nk}y_k = s_n + t_n,
$$

where $s_n = \sum_{k=1}^{\infty} a_{nk}(x_k - y_k)$ and $t_n = \sum_{k=1}^{\infty} a_{nk}y_k$. By applying part (ii), we deduce that $A \in (c_0(E), c_0)$. So $\lim\limits_{n\to\infty} s_n = 0$, since$(x_k - y_k) \in c_0(E)$. On the other hand, $\lim\limits_{n\to\infty} t_n = 0$ by the condition (10), since

$$
t_n \leq \sum_{k=1}^{\infty}\sum_{i\in E_k}|a_{ni}y_i| \leq l\sum_{k=1}^{\infty}\sup_{i\in E_k}|a_{ni}|.
$$

Therefore $A \in (c(E), c_0)$.

(iv) Let $A \in (c_0(E), c)$. Using the sequences e^k, the necessity of the condition (11) is immediate. The proof of the necessity of the condition (8) is similar to the part (i). Conversely, suppose that the conditions (8) and (11) hold. If $B = (b_{ni})$ be a matrix such that $b_{ni} = a_{ni} - l_k$ where $i \in E_k$, for every $n, k \in \mathbb{N}$, we first prove that $B \in (c_0(E), c_0)$. Let $M > 0$ and $\epsilon > 0$ be given. Due to (11), there is a positive number N_k such that $\left|\sup_{i\in E_k} a_{ni} - l_k\right| \le \frac{\epsilon}{M}$, for all $n \ge N_k$. Take $N = \max_{1\le k\le M} N_k$, we have

$$\sum_{k=1}^{m} |l_k| \le \sum_{k=1}^{m} \left| l_k - \sup_{i\in E_k} a_{ni} \right| + \sum_{k=1}^{m} \sup_{i\in E_k} |a_{ni}| \le \epsilon + \sum_{k=1}^{m} \sup_{i\in E_k} |a_{ni}|,$$

for every $n \ge N$. If $M \to \infty$, due to the condition (8), it can be concluded that $(l_k)_{k=1}^{\infty} \in l_1$. Since $\sup_{i\in E_k} b_{ni} = \sup_{i\in E_k} a_{ni} - l_k$, we have $\sup_{i\in E_k} b_{ni} \to 0$ as $n \to \infty$ and also $\sup_n \sum_{k=1}^{\infty} \sup_{i\in E_k} |b_{ni}| < \infty$. Hence $B \in (c_0(E), c_0)$, by part (ii). This implies that $Bx \in c_0$ for all $x \in c_0(E)$, so

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} l_k \left(\sum_{i\in E_k} x_i \right). \tag{13}$$

Since $(\sum_{i\in E_k} x_i)_{k=1}^{\infty} \in c_0$ when $x \in c_0(E)$ and $(l_k)_{k=1}^{\infty} \in c_0^{\beta}$, by Corollary 4, we have $\sum_{k=1}^{\infty} l_k \left(\sum_{i\in E_k} x_i\right) < \infty$ for all $x \in c_0(E)$. This result and the relation (13) show that $B \in (c_0(E), c_0)$. The proof of the part (v) is similar to the part (iii).

It should be noted when $E_j = \{j\}$ for $j = 1, 2, \cdots$, the conditions (8), (9), (10), (11) and (12) can be rewritten as follows, respectively.

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| \right) < \infty, \tag{14}$$

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \cdots), \tag{15}$$

$$\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} a_{nk} \right) = 0, \tag{16}$$

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for some } l_k \in \mathbb{R} \ (k = 1, 2, \cdots), \tag{17}$$

$$\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} a_{nk} \right) = l \quad \text{for some } l \in \mathbb{R}. \tag{18}$$

Corollary 6 [**11**]. We have

(i) $(c_0, l_\infty) = (c, l_\infty) = (l_\infty, l_\infty)$ and $A \in (l_\infty, l_\infty)$ if and only if the condition (14) holds;

(ii) $A \in (c_0, c_0)$ if and only if the conditions (14) and (15) hold ;

(iii) $A \in (c, c_0)$ if and only if the conditions (14), (15) and (16) hold;

(iv) $A \in (c_0, c)$ if and only if the conditions (14) and (17) hold;

(v) $A \in (c, c)$ if and only if the conditions (14), (17) and (18) hold.

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Bəzi yeni qeyri-mütləq tip yarım normallaşmış ardıcıllıq fəzaları və matrix çevrilmələri

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XÜLASƏ

Bu işin məqsədi ardıcıllıq fəzalarını daxil etməkdir. Biz bu fəzaların bəzi topoloji xassələri araşdırırıq və həmçinin onların arasında daxilolma münasibətləri qururuq. Bundan əlavə, biz bu fəzaların α- və β-duallarını hesablayır və sonsuz matrislərin siniflərini xarakterizə edirik.

Açar sözlər: yarımnormallaşmış ardıcıllıq fəzaları, matris oblastı, α- və β-dualları, matris çevirmələr.

Некоторые новые неабсольютного типа полу-нормированные пространства последовательностей и матричные преобразования

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РЕЗЮМЕ

Цель данного исследования заключается в введении пространств последовательностей. Мы исследуем некоторые топологические свойства этих пространств, а также установить некоторые отношений включений между ними. Кроме того, мы вычисляем α- и бета-двойственные этих пространств и охарактеризуем классы бесконечных матриц.

Ключевые слова: полу-нормированные пространства последовательностей, матричные домены, α- и бета-сопряженные, матричные преобразования.