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ESTIMATE OF THE SOLUTION FOR DIFFERENTIAL OPERATOR EQUATION WITH SMALL PARAMETER

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Abstract. In this paper, the representation of the solution to the Cauchy problem and the evolution of this solution in the case of uniqueness are investigated. The problem is considered in a Banach space E. For second-order differential operator equations with a small positive parameter, singular perturbation of the Cauchy problem with an initial condition is examined.

Keywords: Differential equation, singular perturbation, the Cauchy problem, the convergence problem.

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1. Introduction

Abstract the Cauchy problem and its particular cases associated with to differential operator equations play a significant role in functional analysis and in the theory of nonlocal (convolution-type) equations, with a wide range of applications.

In this paper, we study the singular perturbation of the Cauchy problem associated with a second-order differential operator equation containing a small parameter, as well as its limiting first-order form

$$\begin{cases} -\varepsilon u''(t;\varepsilon) + u'(t;\varepsilon) + Au(t;\varepsilon) = f(t;\varepsilon), t \geq 0, \\ u(0;\varepsilon) = u_0(\varepsilon),\ \ u'(0;\varepsilon) = u_1(\varepsilon) \end{cases} \tag{1}$$

and

$$\begin{cases} u'(t) + Au(t) = f(t), t \geq 0, \\ u(0) = u_0 \,. \end{cases} \tag{2}$$

Here, A is a densely defined, self-adjoint and nonnegative operator in a complex Banach space E.

The main goal of our research is to analyze the solution of given problem in the case $\varepsilon \to 0$ and to derive new results concerning the limiting behavior.

In earlier works, similar problems have been investigated using different approaches. For instance, Kisynski [10,11] and other authors [15,16,17] examined the problem in the Hilbert space setting. More general approaches [13,14,18,19] are based on the assumption where A is an infinitesimal generator of the cosine operator function or a strongly continuous semigroup.

In this work, unlike the previous methods, we do not require the initial data u_0 and u_1 to be constant, which ensures that the results remain valid in more general cases. Furthermore, we use the approach based on the integral representations proposed by Dettman [3].

Results in this field have also bees reviewed based on references [1-7,12,20,21] on generalized forms for time-dependent operators. For further information, see Geel's monograph [8].

Let E be a complex space.

If given space E is a complete normed space, then E is a complex Banach space. Let A be a defined operator on this space.

Definition 1.1 If A_n is a sequence of operators in $B(E_1, E_2)$ and

$$\lim_{n\to\propto}\|A_n - A\| = 0$$

for some $A \in B(E_1, E_2)$, then we say that A_n converges uniformly to A or that A_n converges to A in the uniform or operator norm, topology on $B(E_1, E_2)$.

Here, the space $B(E_1, E_2)$denotes the space of all bounded linear operators from $E_1$to E_2. If $E_1 = E_2$, then the space is $B(E, E)$ and we denote it as $B(E)$.

Definition 1.2. A C_0 semigroup or a strongly continuous semigroup in a Banach space $E_1$is a family $A = \big(A(t)\big)_{t\geq 0}$ of bounded linear operators such that:

a) $A(t+s) = A(t)\cdot A(s), \quad t, s \geq 0;$
b) $A(0) = I_X;$
c) $\lim_{t\to 0} A(t)x = x, \;\; x \in E_1.$

If these first two conditions are satisfied, the family is said to be a semigroup.

Definition 1.3. A strongly continuous cosine operator function (or a cosine family) is a family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators satisfying:

a) $C(t+s) + C(t-s) = 2C(t)\cdot C(s), \quad t, s \in \mathbb{R};$
b) $C(0) = I_X;$
c) $\lim_{t\to 0} C(t)x = x, \;\; x \in E_1.$

Suppose that $\{Cos_A(t), t \in \mathbb{R}\}$ is a strongly conditioned cosine family. Then the formula:

$$T(t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} cos_A(s)ds, \qquad t > 0 \tag{3}$$

defines a strongly continuous semigroup whose generator turns out to be A. Relation (3) is known as the Weierstrass formula.

The sine operator function $S(t)$ is defined by the formula

$$S(t) = \int_0^t C(s)ds, \qquad t \in \mathbb{R}. \tag{4}$$

Let be the function f is complex-valued function of the variable $t > 0$ and s is a complex parameter. We define the Laplace transform of f as

$$F(s) = \mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st} f(t)dt$$

whenever the limit exists. When it does, this integral is said to converge. The notation $\mathcal{L}(f)$ will also be used to denote the Laplace transform of f.

The following Proposition 1.1, Corollaries 1.1-1.2 are stated according to the results obtained in [17].

Proposition 1.1. If C is a regular cosine function, then there exist two non-negative constants M and w such that for every $t \in R^{+}$:
$\|C(t)\| \leq Mcosh\omega(t)$.

Corollary 1.1. For $\xi \in R^{+}$:

$$I_1(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{2k+1}}{2^{2k+1}k!\,(k+1)!}.$$

Corollary 1.2. For $\xi \in R$:

$$I_0(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{2^{2k}(k!)^2}.$$

Lemma 1.1. If the function I_0 is an infinite differentiable in the space R, then

$$I_0{}' = I_1.$$

2. Singular perturbation of the Cauchy problem with a small parameter

In this section, we consider the Cauchy problem in the Banach space E for the following non-homogeneous differential operator equation:

$$\begin{cases} -\varepsilon u''(t;\varepsilon) + u'(t;\varepsilon) + Au(t;\varepsilon) = f(t;\varepsilon), t \geq 0, \\ \qquad u(0;\varepsilon) = u_0(\varepsilon),\ \ u\,'(0;\varepsilon) = u_1(\varepsilon). \end{cases} \tag{5}$$

Here, A is a densely defined, nonnegative operator in the Banach space E, and $u(t;\varepsilon)$ is the unknown solution.

To solve eqution (5), let us introduce a new unknown function by making the substitution $u(t;\varepsilon) = e^{t/2\varepsilon} v\left(\frac{t}{\varepsilon};\varepsilon\right)$. Then the Cauchy problem (5) is expressed by the following equation with initial conditions:

$$\begin{cases} -\frac{1}{\varepsilon} v''(t;\varepsilon) + \left(\frac{1}{4\varepsilon} + A\right) v(t;\varepsilon) = e^{-\frac{t}{2\varepsilon}} f(\varepsilon t;\varepsilon), \\ \qquad v(0;\varepsilon) = u_0(\varepsilon), \\ \qquad v'(0;\varepsilon) = \varepsilon u_1(\varepsilon) - \frac{1}{2} u_0(\varepsilon). \end{cases} \tag{6}$$

To find the solution of the problem (6) with the help of the introduced function, let us first consider the case $f(t;\varepsilon) \equiv 0$, that is, in the homogeneous case. It is known that in the homogeneous case the solution is given by

$$v_b(t;\varepsilon) = C(t;\varepsilon)v(0;\varepsilon) + S(t;\varepsilon)v'(0;\varepsilon). \tag{7}$$

In the non-homogeneous case, the general solution of the problem (6), by the method of variation of constants, is written as

$$v(t;\varepsilon) = C(t;\varepsilon)v(0;\varepsilon) + S(t;\varepsilon)v'(0;\varepsilon) +$$
$$+ \int_0^t S(t-s;\varepsilon) e^{-\frac{s}{2\varepsilon}} f(\varepsilon s;\varepsilon) ds \tag{8}$$

Here $C(t;\varepsilon), S(t;\varepsilon)$ are, respectively, called the cosine and sine functions of the operator, and sometimes they are also referred to propagators of the homogeneous equation $v''(t)-\left(A\varepsilon+\frac{1}{4}I\right)v(t)=0$. These functions are defined as follows: $C(t;\varepsilon)v=v(t)$ $v\in D$ and $v(\cdot)$ is the solution satisfying the conditions $v(0)=v, v'(0)=0$.

The cosine operator function $C(t;\varepsilon)$ is continuous on the entire Banach space E, and is strongly continuous for $t\geq 0$. The sine operator function $S(t;\varepsilon)$ is also defined by

$$S(t;\varepsilon)u=\int_0^t C(s;\varepsilon)uds.$$

For thr operator functions $C(t;\varepsilon), S(t;\varepsilon)$, the following decompositions can be written. Namely,

$$u(t;\varepsilon)=C(t)u-\frac{t}{2\sqrt{\varepsilon}}\int_0^t\frac{I_1\left(\frac{(t^2-s^2)^{1/2}}{2\sqrt{\varepsilon}}\right)}{(t^2-s^2)^{1/2}}C(s)uds \tag{9}$$

and

$$S(t;\varepsilon)u=-\int_0^t I_0\left(\frac{(t^2-s^2)^{1/2}}{2\sqrt{\varepsilon}}\right)C(s)uds\,, \qquad (u\in E), t\geq 0. \tag{10}$$

Here, ${I_0}'=I_1$ [17].

Now, using the subtitution $u(t;\varepsilon)=e^{t/2\varepsilon}v\left(\frac{t}{\varepsilon};\varepsilon\right)$ in representation (8), we can write the solution as follows:

$$u(t;\varepsilon)=e^{t/2\varepsilon}C(t;\varepsilon)v(0;\varepsilon)+e^{t/2\varepsilon}S(t;\varepsilon)v'(0;\varepsilon)+$$
$$+e^{t/2\varepsilon}\int_0^t S(t-s;\varepsilon)e^{-\frac{s}{2\varepsilon}}f(\varepsilon s;\varepsilon)ds. \tag{11}$$

If we consider the initial contions $v(0;\varepsilon)=u_0(\varepsilon)$ və $v'(0;\varepsilon)=\varepsilon u_1(\varepsilon)-\frac{1}{2}u_0(\varepsilon)$ in the expression (11) for the solution $u(t;\varepsilon)$, then the general solution of the problem (5) is as

$$u(t;\varepsilon)=e^{t/2\varepsilon}C(t;\varepsilon)u_0(\varepsilon)+e^{t/2\varepsilon}S(t;\varepsilon)\left[\varepsilon u_1(\varepsilon)-\frac{1}{2}u_0(\varepsilon)\right]+$$
$$+e^{t/2\varepsilon}\int_0^t S(t-s;\varepsilon)e^{-\frac{s}{2\varepsilon}}f(\varepsilon s;\varepsilon)ds. \tag{12}$$

Substituting expressions (9) and (10) into the solution of (12), we get that

$$u(t;\varepsilon)$$

$$= e^{\frac{t}{2\varepsilon}} C\left(\frac{t}{\varepsilon}\right) u_0(\varepsilon) - \frac{e^{\frac{t}{2\varepsilon}} t}{2\sqrt{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \frac{I_1\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right)}{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}} C(s) u_0(\varepsilon) ds$$

$$- e^{\frac{t}{2\varepsilon}} \int_0^{\frac{t}{\varepsilon}} I_0\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right) C(s)\left[\varepsilon u_1(\varepsilon) - \frac{1}{2} u_0(\varepsilon)\right] ds$$

$$- e^{\frac{t}{2\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \left(\int_0^{\frac{t}{\varepsilon}-s} I_0\left(\frac{\left((\frac{t}{\varepsilon} - s)^2 - \sigma^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right) C(\sigma) u d\sigma \right) e^{-\frac{s}{2\varepsilon}} f(\varepsilon s;\varepsilon) ds$$

$$= e^{\frac{t}{2\varepsilon}} C\left(\frac{t}{\varepsilon}\right) u_0(\varepsilon) - \frac{e^{\frac{t}{2\varepsilon}} t}{\sqrt{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \frac{I_1\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right)}{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}} C(s) \left(\frac{1}{2} u_0(\varepsilon)\right) ds$$

$$+ e^{\frac{t}{2\varepsilon}} \int_0^{\frac{t}{\varepsilon}} I_0\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right) C(s)\left[\varepsilon u_1(\varepsilon) - \frac{1}{2} u_0(\varepsilon)\right] ds$$

$$- e^{\frac{t}{2\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \left(\int_0^{\frac{t}{\varepsilon}-s} I_0\left(\frac{\left((\frac{t}{\varepsilon} - s)^2 - \sigma^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right) C(\sigma) u d\sigma \right) e^{-\frac{s}{2\varepsilon}} f(\varepsilon s;\varepsilon) ds. \qquad (13)$$

Let us use the following substitutions in solving (13):

1) $\varphi(t,s;\varepsilon) = -\frac{e^{\frac{t}{2\varepsilon}} t}{\sqrt{\varepsilon}} \frac{I_1\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right)}{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}},$

2) $\psi(t,s;\varepsilon) = e^{\frac{t}{2\varepsilon}} I_0\left(\frac{\left(\left(\frac{t}{\varepsilon}\right)^2 - s^2\right)^{1/2}}{2\sqrt{\varepsilon}}\right)$.

Hence, the solution can be written in the form:

$$u(t;\varepsilon) = e^{\frac{t}{2\varepsilon}} C\left(\frac{t}{\varepsilon}\right) u_0(\varepsilon) + \int_0^{\frac{t}{\varepsilon}} \varphi(t,s;\varepsilon)\, C(s)\left(\frac{1}{2}u_0(\varepsilon)\right) ds$$
$$+ \int_0^{\frac{t}{\varepsilon}} \psi(t,s;\varepsilon) C(s)\left[\varepsilon u_1(\varepsilon) - \frac{1}{2}u_0(\varepsilon)\right] ds$$
$$+ \int_0^t \left(\int_0^{\frac{t}{\varepsilon}-s} \psi(t-s,\sigma;\varepsilon) C(\sigma) u d\sigma \right) f(s;\varepsilon) ds. \qquad (14)$$

Thus, the general solution of the problem can be written in a more compact form by introducing the following substitutions:

1) $P(t;\varepsilon) = \int_0^{\frac{t}{\varepsilon}} \varphi(t,s;\varepsilon)\, C(s) ds$;

2) $Q(t;\varepsilon) = \int_0^{\frac{t}{\varepsilon}} \psi(t,s;\varepsilon) C(s) ds$.

Then, from (14) our solution becomes

$$u(t;\varepsilon) = e^{\frac{t}{2\varepsilon}} C\left(\frac{t}{\varepsilon}\right) u_0(\varepsilon) + P(t;\varepsilon)\left(\frac{1}{2}u_0(\varepsilon)\right) + Q(t;\varepsilon)\left[\varepsilon u_1(\varepsilon) - \frac{1}{2}u_0(\varepsilon)\right]$$
$$+ \int_0^t Q(t-s;\varepsilon) f(s;\varepsilon) ds. \qquad (15)$$

Here, the functions $\varphi(t,s;\varepsilon), \psi(t,s;\varepsilon)$ are real-valued functions, $P(t;\varepsilon), Q(t;\varepsilon)$ are operator-valued functions.

From their definitons, it is easy to verify that $P(t;\varepsilon), Q(t;\varepsilon)$ are strongly continuous with respect to t.

Now, using representation (15), we can investigate the problem of estimating the homogeneous solution. Here, for $f(t) = f(t;\varepsilon) = 0$ and $\varepsilon \to 0$, we analyze the convergence $u(t;\varepsilon) \to u(t)$. Firstly, let us assume that the solution $u(t;\varepsilon)$ is uniformly bounded. Then, making use of (15) and setting $\varphi, \psi \geq 0$, we can carry out the estimation process.

To begin, for the cosine and sine operator functions, we show the following inequalities:

$$\|C(t)\| \leq C_0 cosh\omega(t) \qquad -\infty < t < \infty \qquad (16)$$

and

$$\|S(t)\| \leq C_0 e^{\omega^2 t} \qquad (t \geq 0). \qquad (17)$$

From inequality (16), then we obtain:

$$\|u(t;\varepsilon)\| \leq C_0 e^{\frac{t}{2\varepsilon}} \cosh\left(\frac{\omega t}{\varepsilon}\right) \|u_0(\varepsilon)\| + C_0 \int_0^{\frac{t}{\varepsilon}} \varphi(t,s;\varepsilon)\, cosh\omega s \left(\frac{1}{2}\|u_0(\varepsilon)\|\right) ds$$
$$+ C_0 \int_0^{\frac{t}{\varepsilon}} \psi(t,s;\varepsilon)\, cosh\omega s \left(-\frac{1}{2}\|u_0(\varepsilon)\| + \varepsilon\|u_1(\varepsilon)\|\right) ds. \quad (18)$$

By using (4) and (17) we get the following estimation:

$$\|u(t;\varepsilon)\| \leq C_0 e^{\omega^2 t}(\|u_0(\varepsilon)\| + \varepsilon\|u_1(\varepsilon)\|). \quad (19)$$

To investigate the convergence of $u(t;\varepsilon)$, we will make use of the asymptotic expansion of the Bessel function series:

$$I_\vartheta(x) = \frac{e^x}{(2\pi x)^{1/2}} \left(\sum_{j=0}^{m} \frac{(-1)^j[\vartheta, j]}{(2x)^j} + O\left(\frac{1}{x^{m+1}}\right) \right). \quad (20)$$

Here, $[\vartheta, j] = \Gamma\left(\vartheta + j + \frac{1}{2}\right)/j!\,\Gamma\left(\vartheta - j + \frac{1}{2}\right)$.

From the regularity of $x^{-1}I_1(x)$ at x=0, it follows that there exists a constant $C > 0$ such that

$$I_0(x) \leq C(1+x)^{-1/2}e^x, \qquad x^{-1}I_1(x) \leq C(2+x)^{-3/2}e^x \quad (x \geq 0). \quad (21)$$

Now, to estimate $u(t;\varepsilon) - u(t)$, we employ Dettman's method [3]. So, here we will solve by dividing the integration interval of the solution into two subintervals, i.e., $0 \leq s \leq s(\varepsilon), s(\varepsilon) \leq t \leq \frac{t}{\varepsilon}$. Then, we can define the division point for these integrals as follows:

$$\frac{1}{2\sqrt{\varepsilon}} \left(\left(\frac{t}{\varepsilon}\right)^2 - s(\varepsilon)^2 \right)^{1/2} = \frac{\eta t}{\varepsilon}, \quad (22)$$

where $\eta < \frac{1}{2}$. Then the length of the second interval is:

$$\frac{t}{\varepsilon} - s(\varepsilon) = \frac{4\eta^2 t^2}{t + \varepsilon s(\varepsilon)} \leq 4\eta^2 t. \quad (23)$$

If we apply the case m=0 in the interval $0 \leq s \leq s(\varepsilon)$ to the series (20), we obtain by making certain transformations

$$\varphi(t,s;\varepsilon) = \chi(t,s;\varepsilon) exp\left\{ \frac{t}{2\varepsilon} \left(1 + \left(\frac{1}{\varepsilon} - \varepsilon \left(\frac{s}{t}\right)^2 \right)^{1/2} \right) \right\}. \quad (24)$$

Here, when $\frac{\varepsilon}{t} \to 0$, then

$$\chi(t,s;\varepsilon) = (\pi t)^{-1/2}\sqrt{\varepsilon} \left(\frac{1}{\varepsilon} - \varepsilon\left(\frac{s}{t}\right)^2 \right)^{-1/4} \left(1 + O\left(\frac{\varepsilon}{\eta t}\right) \right).$$

is obtained. Using inequality (21), let us estimate the function φ in the interval $s(\varepsilon) \leq t \leq \frac{t}{\varepsilon}$.

So that,

$$\varphi(t,s;\varepsilon) \le C\rho(t,s;\varepsilon)exp\left\{\frac{t}{2\varepsilon}\left(1+\left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2}\right)\right\} \qquad (25)$$

where

$$\rho(t,s;\varepsilon) = t^{-1/2}\left(\frac{4\varepsilon}{t}+\left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2}\right)^{-1/2}.$$

Now, applying (25), we evaluate the integral for $s(\varepsilon) \le t \le \frac{t}{\varepsilon}$. Since $\left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2} = 2\eta$ and $\frac{4\varepsilon}{t}$, we obtain

$$\varphi(t,s;\varepsilon) \le Ct^{-1/2}\eta^{-1/2}\cdot e^{\frac{t}{2\varepsilon}}\cdot e^{\frac{\eta t}{\varepsilon}}. \qquad (26)$$

Hence,

$$\int_{s(\varepsilon)}^{\frac{t}{\varepsilon}} \varphi(t,s;\varepsilon)\|C(s)u\|\,ds \le C\eta^{1/2}\|u\|\frac{t^{1/2}}{\varepsilon}exp\left\{\frac{t}{2\varepsilon}\left(\frac{1}{2}-\eta-\omega\varepsilon\right)\right\}. \qquad (27)$$

To compute the integral, we will use expression (24). Since $\left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2} \le 1-\frac{1}{2}\varepsilon\left(\frac{s}{t}\right)^2$, the exponential term can be bounded above by $-s^2/4t$. In the denominator, however, $\left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2} \ge \left(\frac{1}{\varepsilon}-\varepsilon\left(\frac{s(\varepsilon)}{t}\right)^2\right)^{1/2} = 2\eta$, which yields the following estimate:

$$\varphi(t,s;\varepsilon) \le Ct^{-1/2}\eta^{-1/2}\cdot e^{-s^2/4t}. \qquad (28)$$

Here, C is independent of ε.

We prove below that for $\frac{t(\varepsilon)}{\varepsilon} \to \infty$ $(\varepsilon \to 0)$ the expression $P(t;\varepsilon)u \to S(t)u$ converges regularly with respect to $u \in E$ in bounded subspaces of E. Suppose the opposite. If our proposition is not satisfied, then there is a bounded sequence $\{u_n\} \subset E$, a sequence $\{t_n\}$ and a sequence $\{\varepsilon_n\}$ as $\varepsilon_n \to 0$ such that

$$\frac{t_n}{\varepsilon_n} \to \infty \qquad (\cdot)$$

and the following estimation is true:
$\|P(t_n;\varepsilon_n)u_n - S(t_n)u_n\| \ge \delta > 0.$

For each n, we choose η_n so that

$$\frac{1}{2}-\eta_n-\omega\varepsilon_n = \frac{\varepsilon_n}{t_n^{1/2}} \qquad (29)$$

and when $\eta = \eta_n$, the integration interval is split according to the expression (22). Then we obtain

$$\|P(t_n;\varepsilon_n)u_n - S(t_n)u_n\| \le \int_0^{s(\varepsilon_n)} \left|\varphi(t_n,s;\varepsilon_n)-(\pi t_n)^{-1/2}e^{-\frac{s^2}{4t_n}}\right|\|C(s)u_n\|ds +$$

$$+\int_{s(\varepsilon_n)}^{\frac{t_n}{\varepsilon_n}} \varphi(t_n, s; \varepsilon_n)\|C(s)u_n\|\, ds +$$

$$+(\pi t_n)^{-1/2} \int_{s(\varepsilon_n)}^{\infty} e^{-\frac{s^2}{4t_n}}\|C(s)u_n\|\, ds\,. \tag{30}$$

The first integral tends to zero as $n \to \infty$, the second integral also tends to zero by virtue of inequality (27). It is shown that the third integral in expression (30) tends to zero as $n \to \infty$. By introducing the subtitution $t_n^{-1/2}s = \sigma$, the integral is reduced to a Gauss-type form. Using the boundedness of $\{u_n\}$ and the exponential estimate for $C(s)$, one obtains uniform upper bound for the integrand. The quadratic term in the exponent dominates any possible exponential growth, ensuring that the tail of the integral vanishes. Consequently, the integral converges to zero as $n \to \infty$. Hence, we get a contradiction. Therefore, our claim is fulfilled.

Similarly, the convergence of the operator function $Q(t;\varepsilon)$ can also be proved. Indeed, these estimates can be verified in the same way as follows:

a) From expression (24), we obtain

$$\chi(t,s;\varepsilon) = (\pi t)^{-1/2}\sqrt{\varepsilon}\left(\frac{1}{\varepsilon} - \varepsilon\left(\frac{s}{t}\right)^2\right)^{1/4}\left(1 + O\left(\frac{\varepsilon}{\eta t}\right)\right). \tag{31}$$

b) In the interval $s(\varepsilon) \le t \le \frac{t}{\varepsilon}$, from the estimate (25) we obtain that

$$\rho(t,s;\varepsilon) = t^{-1/2}\left(\frac{2\varepsilon}{t} + \left(\frac{1}{\varepsilon} + \varepsilon\left(\frac{s}{t}\right)^2\right)^{1/2}\right)^{-1/2}.$$

c) For the interval $s(\varepsilon) \le t \le \frac{t}{\varepsilon}$, we have

$$\psi(t,s;\varepsilon) \le Ct^{-1/2}\eta^{1/2} \cdot e^{\frac{t}{2\varepsilon}} \cdot e^{\frac{\eta t}{\varepsilon}}. \tag{32}$$

Thus, this expression is satisfied:

$$\int_{s(\varepsilon)}^{\frac{t}{\varepsilon}} \psi(t,s;\varepsilon)\|C(s)u\|\, ds \le C\eta^{5/2}\|u\|\frac{t^{1/2}}{\varepsilon} exp\left\{\frac{t}{2\varepsilon}\left(\frac{1}{2} - \eta - \omega\varepsilon\right)\right\}. \tag{33}$$

d) For $0 \le s \le s(\varepsilon)$, the function ψ satisfies the estimation (28).

Using these clauses, we can prove in the same way that

$$Q(t;\varepsilon)u \to S(t)u. \tag{34}$$

So this convergence (34) is regular.

If we take all these estimates into calculation in solving problem (15), we arrive at the following theorem:

Theorem 2.1. Suppose that $u_0(\varepsilon), u_1(\varepsilon) \in E$. Thus,

$$u_0(\varepsilon) \to v, \varepsilon u_1(\varepsilon) \to u_0 - v \quad (\varepsilon \to 0). \tag{35}$$

Assume that $u(t;\varepsilon)$ is the generalized solution of the equation (5) and the condition (.) is satisfied in the case $t(\varepsilon) > 0$. Then the following expression is convergent in $t \ge t(\varepsilon)$;

$$u(t;\varepsilon) \to u(t). \quad (36)$$

Here, the function $u(t)$ is a generalized solution of the equation (2). If $\|u_0\|, \|v\|$ are bounded, then this convergence is uniform with respect to u_0, v.

3. Conclusion

In this work, the existence of the solution to the Cauchy problem for given differential equation with small positive parameter has been rigorously investigated and proven. Moreover, the estimates have been derived for this solution in the homogeneous case. The obtained results are of great scientific importance in terms of studying the effect of a small parameter on the solution of the system and contribute to the theory of solutions of differential operator equations.

References

1. Ashyralyev A., Cuevas C., and Piskarev S. "On well-posedness of difference schemes for abstract elliptic problems in L^p ([0,T];E) spaces", Numer. Funct. Anal. Optim., V.29, (2008), pp.43-65.
2. De Jager E. M. Singular perturbations of hyperbolic type, Bieuw Arch. Wisk., V.23, (1975), pp.145-171.
3. Dettman J. W. Perturbation techniques for related differential equations. J. Differential Equations, V.14, (1973), pp.547-558.
4. Dore G. and Yakubov S. Semigroup estimates and noncoercive boundary value problems, Semigroup Forum, V.60, (2000), pp.93-121.
5. Fattorini H.O. Second Order Linear Differential Equations in Banach Spaces, Notas de Matematica, Elseiver-North Holland, V.99, (1985).
6. Fattorini H.O. Singular Perturbation and Boundary Layer for an Abstract Cauchy Problem, Journal of Math. Analy. and Appl., V.97, (1983), pp.529-571.
7. Friedman A. Singular perturbation for the Cauchy problem and for boundary value problems, J. Differential Equations, V.5, (1969), pp.226-261.
8. Geel R. Singular Perturbations of Hyperbolic Type, Mathematical Centre Tracts., N.98, Mathematisch Centrum, Amsterdam, (1978).
9. Genet J. and Madaune M. Perturbations singulieres pour une classe de problemes hyperboliques non lineaires, Lecture Notes in Mathematics, Springer, Berlin, V. 594, (1977), pp.201-231.
10. Kisynski J. Sur les equations hyperboliques avec petit parametre, Collect. Math., V.10, (1963), pp.331-343.
11. Kisynski J. On second order Cauchy's problem in a Banach space, Bull. Acad. Polon. Sci. Ser. Sci. Math., V.VIII, (1970), pp.371-374.
12. Krein S.G. Linear Differential Equations in Banach Spaces, Izdat. Nauka, Moscow, 1967, English translation: Trans. Math. Monographs, Vol. 29, American Mathematical Society, Providence, RI, 1971.
13. Musaev H.K. and Shakhmurov V.B. Regularityproperties of degenerate convolution-elliptic equations, Boundary Value Probl., V.2016, (2016),

pp.1-19.
14. Nur H.S. Singular perturbations of differential equations in abstract spaces, Pacific J. Math., V.36, (1971), pp.775-780.
15. Prüss J. Evolutionary Integral Equations and Applications, Monographs in Mathematics, V.87, (1993).
16. Sova M. Cosine operator functions, Rozprawy Mat., V.49, (1966), pp.1-47.
17. Sova M. Perturbations numeriques des evolutions parabolique et hyperbolique, Casopis Pest. Mat., V.96, (1971), pp.406-425.
18. Yakubov S. and Yakubov Ya. Differential-Operator Equations: Ordinary and Partial Differential Equations, Monographs and Surveys in Pure and Applied Mathematics, V.103, (2000).
19. Shakhmurov V.B. and Musaev H.K. Separability properties of convolution-differential operator equations in weighted L_p spaces, Appl. Comput. Math., V.14, (2015), pp.221-233.
20. Shakhmurov V. and Shakhmurov R. Maximal B-regular integro-differential equation, Chin. Ann. Math., Ser. B., V.30, (2008), pp.39-50.
21. Weis L. Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann., V.319, (2001), pp.735-758.