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INVESTIGATION OF SOLUTIONS TO BOUNDARY VALUE AND STEKLOV PROBLEMS FOR AN ORDINARY DIFFERENTIAL EQUATION WITH FRACTIONAL-ORDER DERIVATIVES

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Abstract. In the present work, boundary and Steklov problems are considered for a linear ordinary differential equation with constant coefficients and fractional-order derivatives. An analytical expression is obtained for the classical solution of the boundary value problem. The general solution of the equation is defined (expressed) in the form of the Mittag–Leffler function. For the Steklov problem, the eigenvalue and eigenfunction are determined..

Keywords: Fractional-order derivative, boundary conditions, boundary value problem, spectral parameter, Steklov problem, general solution, classical solution, eigenvalue, eigenfunction.

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Introduction.

It is well known that boundary value problems for ordinary differential equations of first to fourth order derivatives have been extensively studied [6, 8]. Since various definitions of fractional-order derivatives have a global character (as they are defined via integrals), the investigation of solutions to the corresponding problems presents significant difficulties. The most commonly used fractional derivatives are the Riemann–Liouville and Caputo derivatives [7]. In this work, we consider a boundary value problem and a Steklov problem for a linear, single-order ordinary differential equation with constant coefficients involving a fractional derivative in the Riemann–Liouville sense. An analytical expression is obtained for the classical solution of the boundary value problem, and the eigenvalue and eigenfunction are determined for the Steklov problem.

The problem for a fractional-order differential equation with a delay argument, the Cauchy problem for differential equations with Caputo fractional derivatives [3, 4, 9], the identification of fractional derivatives in oscillatory systems, the boundary value problem for fractional-order ordinary differential equations, and boundary value problems for partial differential equations with real-order derivatives have been considered in [1, 2, 5].

Statement of the Probelm.

Let us consider the following equation:

$$D^{\frac{3}{4}}y(x) - 2pD^{\frac{2}{4}}y(x) - p^2D^{\frac{1}{4}}y(x) + 2p^3y(x) = 0, \quad 0 < a < x < b \qquad (1)$$

where p is a given real constant, $y(x)$ is the unknown (sought) function. The given equation (1) is a linear homogeneous equation with step size $\frac{1}{4}$ and order $\frac{3}{4}$, constant coefficients. The interval $[a,b]$, which is the closure of its domain of definition, does not contain zero. As mentioned above, the solution of this equation will be sought in terms of a Mittag–Leffler series [7].

$$y(x) \equiv y(x,\rho) = \sum_{k=0}^{\infty} \rho^k \frac{x^{-1+\frac{k+1}{4}}}{\Gamma\left(\frac{k+1}{4}\right)} \qquad (2)$$

here, ρ denotes an arbitrary parameter.

The $\frac{1}{4}$ order derivative of the first term of this series gives the Dirac delta function. As the point at which the argument of this function becomes zero does not lie within $[a,b]$, the function is equal to zero. Therefore, it is easy to see that

$$D^{\frac{n}{4}}y(x) = D^{\frac{n}{4}}y(x,\rho) = \rho^n y(x,\rho), \quad n = 1,2,3 \qquad (3)$$

Considering (2) and (3) in (1) we obtain the algebraic (characteristic) equation:

$$\rho^3 - 2p\rho^2 - p^2\rho + 2p^3 = 0, \qquad (4)$$

The roots of this algebraic equation [5] are the real numbers:

$$\rho_1 = -p, \quad \rho_2 = p, \quad \rho_3 = 2p \qquad (5)$$

Therefore, the general solution to equation (1) is as follows:

$$y(x) = \sum_{n=1}^{3} C_n y(x,\rho_n) = C_1 y(x,-p) + C_2 y(x,p) + C_3 y(x,2p), \quad (6)$$

where C_1, C_2 and C_3 are arbitrary constants.

Now let's consider the following boundary conditions for equation (1):

$$\begin{cases} y(b) - y(a) = \alpha_1, \\ D^{\frac{1}{4}}y(x)\Big|_{x=b} - D^{\frac{1}{4}}y(x)\Big|_{x=a} = \alpha_2, \\ D^{\frac{2}{4}}y(x)\Big|_{x=b} - D^{\frac{2}{4}}y(x)\Big|_{x=a} = \alpha_3, \end{cases} \qquad (7)$$

where α_1, α_2 and α_3 are given real constants.

Main results.

Theorem 1. If p, α_1, α_2 and α_3 are given real constants with $0 < a < b$, then for different ρ_1, ρ_2 and ρ_3 defined as in (5), and if conditions given in (9) are satisfied, then the boundary value problem (1) and (7) has a unique solution, and this solution is given as in (16).

Proof. Substituting the general solution of (6) into the boundary conditions (7) to determine the constants C_1, C_2 and C_3, we obtain a system of inhomogeneous linear algebraic equations:

$$\begin{cases} \sum_{n=1}^{3} C_n[y(b,\rho_n) - y(a,\rho_n)] = \alpha_1, \\ \sum_{n=1}^{3} C_n\,\rho_n[y(b,\rho_n) - y(a,\rho_n)] = \alpha_2, \\ \sum_{n=1}^{3} C_n\rho_n^2[y(b,\rho_n) - y(a,\rho_n)] = \alpha_3, \end{cases} \tag{8}$$

If the condition

$$y_n \equiv y(b,\rho_n) - y(a,\rho_n) \neq 0, \quad n = 1,2,3 \tag{9}$$

is satisfied, then the determinant of the system of inhomogeneous linear algebraic equations (8) is as follows:

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 \\ \rho_1 y_1 & \rho_2 y_2 & \rho_3 y_3 \\ \rho_1^2 y_1 & \rho_2^2 y_2 & \rho_3^2 y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_1^2 & \rho_2^2 & \rho_3^2 \end{vmatrix} y_1 y_2 y_3$$
$$= (\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_3 - \rho_2) y_1 y_2 y_3 \neq 0, \tag{10}$$

Then, we get from the system (8) following expressions:

$$C_1 = \frac{1}{\Delta}\begin{vmatrix} \alpha_1 & y_2 & y_3 \\ \alpha_2 & \rho_2 y_2 & \rho_3 y_3 \\ \alpha_3 & \rho_2^2 y_2 & \rho_3^2 y_3 \end{vmatrix} = \frac{y_2 y_3}{\Delta}\begin{vmatrix} \alpha_1 & 1 & 1 \\ \alpha_2 & \rho_2 & \rho_3 \\ \alpha_3 & \rho_2^2 & \rho_3^2 \end{vmatrix} =$$

$$= \frac{y_2 y_3(\rho_3 - \rho_2)}{(\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_3 - \rho_2) y_1 y_2 y_3}[\alpha_1\rho_2\rho_3 - \alpha_2(\rho_2 + \rho_3) + \alpha_3] =$$

$$= \frac{\alpha_1\rho_2\rho_3 - \alpha_2(\rho_2 + \rho_3) + \alpha_3}{(\rho_2 - \rho_1)(\rho_3 - \rho_1) y_1}, \tag{11}$$

$$C_2 = \frac{1}{\Delta}\begin{vmatrix} y_1 & \alpha_1 & y_3 \\ \rho_1 y_1 & \alpha_2 & \rho_3 y_3 \\ \rho_1^2 y_1 & \alpha_3 & \rho_3^2 y_3 \end{vmatrix} = \frac{y_1 y_3}{\Delta}\begin{vmatrix} 1 & \alpha_1 & 1 \\ \rho_1 & \alpha_2 & \rho_3 \\ \rho_1^2 & \alpha_3 & \rho_3^2 \end{vmatrix} =$$
$$= \frac{-y_1 y_3(\rho_3 - \rho_1)}{(\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_3 - \rho_2) y_1 y_2 y_3}[\alpha_1\rho_1\rho_3 - \alpha_2(\rho_1 + \rho_3) + \alpha_3] =$$
$$= \frac{\alpha_1\rho_1\rho_3 - \alpha_2(\rho_1 + \rho_3) + \alpha_3}{(\rho_1 - \rho_2)(\rho_3 - \rho_1) y_2}, \tag{12}$$

$$C_3 = \frac{1}{\Delta}\begin{vmatrix} y_1 & y_3 & \alpha_1 \\ \rho_1 y_1 & \rho_2 y_2 & \alpha_2 \\ \rho_1^2 y_1 & \rho_2^2 y_3 & \alpha_3 \end{vmatrix} = \frac{y_1 y_2}{\Delta}\begin{vmatrix} 1 & 1 & \alpha_1 \\ \rho_1 & \rho_2 & \alpha_2 \\ \rho_1^2 & \rho_2^2 & \alpha_3 \end{vmatrix} =$$

$$= \frac{y_1 y_2(\rho_2 - \rho_1)}{(\rho_2 - \rho_1)(\rho_3 - \rho_1)(\rho_3 - \rho_2) y_1 y_2 y_3}[\alpha_1\rho_1\rho_2 - \alpha_2(\rho_1 + \rho_2) + \alpha_3] =$$

$$= \frac{\alpha_1\rho_1\rho_2 - \alpha_2(\rho_1+\rho_2)+\alpha_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)y_3}. \qquad (13)$$

By unifying the expressions (11)–(13), we can obtain the following expression for the solution of system (8):

$$C_k = \frac{\alpha_1 P_k - \alpha_2 Q_k + \alpha_3}{y_k R_k}, \quad k = 1,2,3 \qquad (14)$$

here

$$P_k = \prod_{\substack{m=1 \\ m\neq k}}^{3} \rho_m, \; Q_k = \sum_{\substack{m=1 \\ m\neq k}}^{3} \rho_m, \; R_k = \prod_{\substack{m=1 \\ m\neq k}}^{3} (\rho_m - \rho_k). \qquad (15)$$

Then, the classical solution of the boundary value problem (1), (7) will be in the form:

$$y(x) = \sum_{k=1}^{3} \frac{\alpha_1 P_k - \alpha_2 Q_k + \alpha_3}{y_k R_k} y(x, \rho_k), \qquad (16)$$

here ρ_k, $k = \overline{1,3}$ are in the form (5), and P_k, Q_k and R_k $k = \overline{1,3}$ are in the condition (15).

Finally, let's formulate the Steklov problem for equation (1). Let us consider the following boundary conditions:

$$D^{\frac{1}{4}} y(x)\Big|_{x=a} = 0, D^{\frac{1}{4}} y(x)\Big|_{x=b} = 0, y(b) - \lambda y(a) = 0. \qquad (17)$$

Thus, we obtain the Steklov problem (1) and (17). Here, λ is the spectral parameter.

If we substitute (6), which is the general solution of equation (1), into (17), we obtain:

$$\begin{cases} \sum_{n=1}^{3} C_n \rho_n y(a, \rho_n) = 0, \\ \sum_{n=1}^{3} C_n \; \rho_n y(b, \rho_n) = 0, \\ \sum_{n=1}^{3} C_n [y(b, \rho_n) - \lambda y(a, \rho_n)] = 0. \end{cases} \qquad (18)$$

For the existence of a nontrivial solution of the obtained system of homogeneous linear algebraic equations, the determinant must be zero.

$$\Delta = \begin{vmatrix} \rho_1 y(a,\rho_1) & \rho_2 y(a,\rho_2) & \rho_3 y(a,\rho_3) \\ \rho_1 y(b,\rho_1) & \rho_2 y(b,\rho_2) & \rho_3 y(b,\rho_3) \\ y(b,\rho_1) - \lambda y(a,\rho_1) & y(b,\rho_2) - \lambda y(a,\rho_2) & y(b,\rho_3) - \lambda y(a,\rho_3) \end{vmatrix} = 0. \quad (19)$$

From equation (19), we obtain the eigenvalue (for the Steklov problem):

$$\begin{vmatrix} \rho_1 y(a,\rho_1) & \rho_2 y(a,\rho_2) & \rho_3 y(a,\rho_3) \\ \rho_1 y(b,\rho_1) & \rho_2 y(b,\rho_2) & \rho_3 y(b,\rho_3) \\ y(b,\rho_1) & y(b,\rho_2) & y(b,\rho_3) \end{vmatrix} = \lambda \begin{vmatrix} \rho_1 y(a,\rho_1) & \rho_2 y(a,\rho_2) & \rho_3 y(a,\rho_3) \\ \rho_1 y(b,\rho_1) & \rho_2 y(b,\rho_2) & \rho_3 y(b,\rho_3) \\ y(a,\rho_1) & y(a,\rho_2) & y(a,\rho_3) \end{vmatrix},$$

$$\begin{aligned}
&\rho_1\rho_2 y(a,\rho_1)y(b,\rho_2)y(b,\rho_3) + \rho_2\rho_3 y(a,\rho_2)y(b,\rho_3)y(b,\rho_1) + \\
&+\rho_1\rho_3 y(b,\rho_1)y(a,\rho_3)y(b,\rho_2) - \rho_2\rho_3 y(b,\rho_1)y(b,\rho_2)y(a,\rho_3) - \\
&-\rho_1\rho_3 y(b,\rho_2)y(b,\rho_3)y(a,\rho_1) - \rho_1\rho_2 y(b,\rho_1)y(a,\rho_2)y(b,\rho_3) = \\
&= \lambda[\rho_1\rho_2 y(a,\rho_1)y(b,\rho_2)y(a,\rho_3) + \rho_2\rho_3 y(a,\rho_2)y(b,\rho_3)y(a,\rho_1) + \\
&+\rho_1\rho_3 y(b,\rho_1)y(a,\rho_3)y(a,\rho_2) - \rho_2\rho_3 y(a,\rho_1)y(b,\rho_2)y(a,\rho_3) - \\
&-\rho_1\rho_2 y(b,\rho_1)y(a,\rho_2)y(a,\rho_3) - \rho_1\rho_3 y(a,\rho_1)y(a,\rho_2)y(b,\rho_3)],
\end{aligned}$$

$$\begin{aligned}
&\rho_1\rho_2 y(b,\rho_3)[y(a,\rho_1)y(b,\rho_2) - y(b,\rho_1)y(a,\rho_2)] + \\
&+\rho_2\rho_3 y(b,\rho_1)[y(a,\rho_2)y(b,\rho_3) - y(b,\rho_2)y(a,\rho_3)] + \\
&+\rho_1\rho_3 y(b,\rho_2)[y(b,\rho_1)y(a,\rho_3) - y(a,\rho_1)y(b,\rho_3)] = \\
&= \lambda\{\rho_1\rho_2 y(a,\rho_3)[y(a,\rho_1)y(b,\rho_2) - y(b,\rho_1)y(a,\rho_2)] + \\
&+\rho_2\rho_3 y(a,\rho_1)[y(a,\rho_2)y(b,\rho_3) - y(b,\rho_2)y(a,\rho_3)] + \\
&+\rho_1\rho_3 y(a,\rho_2)[y(b,\rho_1)y(a,\rho_3) - y(a,\rho_1)y(b,\rho_3)]\},
\end{aligned}$$

$$\lambda = \frac{y(b,\rho_1)M(\rho_2,\rho_3) - y(b,\rho_2)M(\rho_1,\rho_3) + y(b,\rho_3)M(\rho_1,\rho_2)}{y(a,\rho_1)M(\rho_2,\rho_3) - y(a,\rho_2)M(\rho_1,\rho_3) + y(a,\rho_3)M(\rho_1,\rho_2)}, \quad (20)$$

here

$$M(\rho_m,\rho_n) = \rho_m\rho_n[y(a,\rho_m)y(b,\rho_n) - y(b,\rho_m)y(a,\rho_n)]. \quad (21)$$

The eigenfunction should be obtained from the general solution of equation (6), provided that C_1, C_2 and C_3 are determined from system (18). Since the determinant of system (18) is zero, they represent a linearly dependent system. Therefore, one of the equations in this system can be disregarded, i.e., if we omit the last equation we get system (22):

$$\begin{cases} C_1\rho_1 y(a,\rho_1) + C_2\rho_2 y(a,\rho_2) = -C_3\rho_3 y(a,\rho_3), \\ C_1\rho_1 y(b,\rho_1) + C_2\rho_2 y(b,\rho_2) = -C_3\rho_3 y(b,\rho_3) \end{cases} \quad (22)$$

From this system, under the following condition

$$W=\begin{vmatrix}\rho_1 y(a,\rho_1) & \rho_2 y(a,\rho_2)\\ \rho_1 y(b,\rho_1) & \rho_2 y(b,\rho_2)\end{vmatrix}\neq 0, \qquad (23)$$

we obtain:

$$\begin{cases} C_1=\dfrac{\rho_3}{\rho_1}\dfrac{\begin{vmatrix} y(a,\rho_2) & y(a,\rho_3)\\ y(b,\rho_2) & y(b,\rho_3)\end{vmatrix}}{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_2)\\ y(b,\rho_1) & y(b,\rho_2)\end{vmatrix}}C_3,\\ C_2=\dfrac{-\rho_3}{\rho_2}\dfrac{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_3)\\ y(b,\rho_1) & y(b,\rho_3)\end{vmatrix}}{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_2)\\ y(b,\rho_1) & y(b,\rho_2)\end{vmatrix}}C_3. \end{cases} \qquad (24)$$

Then from (6) we get:

$$(x)=\left\{\frac{\rho_3}{\rho_1}\frac{\begin{vmatrix} y(a,\rho_2) & y(a,\rho_3)\\ y(b,\rho_2) & y(b,\rho_3)\end{vmatrix}}{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_2)\\ y(b,\rho_1) & y(b,\rho_2)\end{vmatrix}}y(x,\rho_1)-\right.$$

$$\left.-\frac{\rho_3}{\rho_2}\frac{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_3)\\ y(b,\rho_1) & y(b,\rho_3)\end{vmatrix}}{\begin{vmatrix} y(a,\rho_1) & y(a,\rho_2)\\ y(b,\rho_1) & y(b,\rho_2)\end{vmatrix}}y(x,\rho_2)+y(x,\rho_3)\right\}C_3, \qquad (25)$$

Thus, we obtain the following results.

Theorem 2. If p is a given real constant and $0<a<b$, then there exists an eigenvalue (spectrum) given in the form (20) of the Steklov problem (1), (17), and the corresponding Eigen function in the form (25). Here, C_3 is an arbitrary constant.

Note. By choosing the constant C_3,, the Eigen function (25) can be normalized. That is

$$\|y(x)\|^2=\int_a^b y^2(x)dx=1.$$

Conclusion

In this work, the boundary value problem and the Steklov problem for a linear homogeneous ordinary differential equation with a fractional-order derivative are investigated. The general solution of the equation is constructed with the help of the Mittag–Leffler function. The arbitrary constants involved in the general solution are determined from the given boundary conditions; as a result, the solution of the boundary value problem, as well as the eigenvalues and eigenfunctions of the Steklov problem, are obtained.

References

1. Abdeljawad T., Shah K., Abdo M.S., Jarad F., An analytical study of fractional delay impulsive implicit sistems with Mittaq-Leffler law, Applied and Computational Mathematics, Vol.22, No.1, 2023, pp.31-44.
2. Agarwal G., Yadav L.K., Nisar K.S., Algarni M.M., Mannoud E.E., A hybrid method for the analytical solution of time fractional Whitham-Broqz-Kaup equation, *Appl. Comput. Math.*, Vol.23, No.1, 2024, pp.3-17.
3. Aliev F.A., Aliev N.A., Mutallimov M.M., Namazov A.A.. Identification method for defining the order of the fractional derivative oscillatory system, *Proceeding of IAM*, Vol.8, No.1, 2019, pp.3-13. (in Russian)
4. Aliyev N.A., Pashavand A.A., A boundary value problem for a fractional order ordinary linear differential equation with a constant coefficient, *Proceeding of IAM*, Vol.4, No.1, 2015, pp.3-7.
5. Bronstein N.N., Semendyaev K.A. *Handbook of Mathematics for Engineers and University Students*, Nauka, Moscow, 1964, 608 p. (in Russian)
6. Khruslov E.Ya., Inverse scattering problem for the electrical prospecting equation, general theory of boundary value problems, Kyiv, Naukova Dumki, 1983, *Collection of Scientific Papers*, pp. 213-219. (in Russian)
7. Kilbas A.A., Srivastava H.M., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, Mathematics Studies, 2004, 524 p.
8. Naimark M.A., *Linear Differential Operators*, Nauka, Moscow, 1959, 528 p. (in Russian)
9. Pashavand A.A., Aliyev N.A., The boundary value problem for real order partial differential equation in the first quarter, *Casp. J. Appl. Math. Ecol. Econ*, Vol.3, No.1, 2015, pp.69-71.