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INVERSE PROBLEM OF RESTORING AN UNKNOWN COEFFICIENT IN A PSEUDOHYPERBOLIC EQUATION

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Abstract. An inverse boundary value problem for a third-order pseudo-hyperbolic equation with integral condition of the first kind is investigated. To study the solvability of the inverse problem, we first reduce the considered problem to an auxiliary problem and prove its equivalence (in a certain sense) to the original problem. Then using the Banach fixed point principle, the existence and uniqueness of a solution to this problem is shown. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the inverse coefficient problem is proved for the smaller value of time.

Keywords: inverse problem, pseudo hyperbolic equation, overdetermination condition, classical solution, existence, uniqueness.

AMS Subject Classification: Primary 35R30, 35L10; Secondary 35A01, 35A02, 35A09.

1.Introduction.

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics.

The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science. Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of A.N.Tikhonov [1], M.M.Lavrentyev [2,3], A.M.Denisov [4], M.I.Ivanchov [5] and their followers.

Contemporary problems of natural sciences make necessary to state and investigate qualitative new problems, the striking example of which is the class of non-local problems for partial differential equations. Among non-local problems we can distinguish a class of problems with integral conditions. Such conditions appear by mathematical simulation of phenomena related to physical plasma [6], distribution of the heat [7] process of moisture transfer in capillary simple environments [8], with the problems of demography and mathematical biology.

The solvability of inverse problems in certain formulations, with certain overdetermination conditions for pseudo-hyperbolic equations, was the subject of research in [9–12].

In this paper , using the Fourier method and the principle of contraction mappings , the existence and uniqueness of a solution to a nonlinear inverse boundary value problem for a pseudo - hyperbolic equation of the third order with an integral condition of the first kind are proved .

2. Problem statement and its reduction to equivalent problem.

Lets consider for the equation

$$(pt+q)^2 u_{tt}(x,t) - \alpha(pt+q)u_{txx}(x,t) - \beta u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \qquad (1)$$

in the domain $D_T = \{(x,t): 0 < x < 1,\ 0 \le t \le T\}$ an inverse boundary problem with initial conditions

$$u(x,0) = \varphi(x),\ \ u_t(x,0) = \psi(x) \ \ (0 \le x \le 1)\,, \qquad (2)$$

periodic condition

$$u(0,t) = u(1,t)\ \ (0 \le t \le T)\,, \qquad (3)$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0\ \ (0 \le t \le T), \qquad (4)$$

and the additional conditions

$$u(x_0,t) = h(t) \quad (0 \le t \le T)\,, \qquad (5)$$

where $p > 0, q > 0, \alpha > 0, \beta > 0$ $x_0 \in (0,1)$ is a fixed numbers, $f(x,t)$, $\varphi(x), \psi(x)$, $h(t)$ are given functions, $u(x,t)$, $a(t)$ and $b(t)$ are unknown functions.

Definition. *The pair* $\{u(x,t), a(t)\}$ *is said to be a classical solution to the problem (1)-(5), if the functions* $u(x,t) \in \tilde{C}^{2,2}(\bar{D}_T)$ *and* $a(t) \in C[0,T]$ *satisfies an Equation (1) in the region* D_T, *the condition (2) on* $[0,1]$, *and the conditions (3)-(5) on* $[0,T]$, *where*

$$\tilde{C}^{(2,2)}(\bar{D}_T) = \{u(x,t): u(x,t) \in C^2(\bar{D}_T), u_{txx}(x,t) \in C(\bar{D}_T)\}.$$

Theorem 1. Suppose that $f(x,t) \in C(\bar{D}_T), \varphi(x) \in C^1[0,1], \psi(x) \in C[0,1]$, $\varphi'(1) - \varphi'(0) = 0$, $h(t) \in C^2[0,T]$, $h(t) \ne 0$ $(0 \le t \le T)$, $\int_0^1 f(x,t)dx = 0$, $(0 \le t \le T)$ and the compatibility conditions

$$\int_0^1 \varphi(x)dx = 0,\ \ \int_0^1 \psi(x)dx = 0\,, \qquad (6)$$

$$\phi(x_0) = h(0),\ \psi(x_0) = h'(0) \qquad (7)$$

holds. Then the problem of finding a classical solution of (1)-(5) is equivalent to the problem of determining functions $u(x,t) \in \tilde{C}^{2,2}(\overline{D}_T)$ and $a(t) \in C[0,T]$, satisfying equation (1), conditions (2) and (3), and the conditions

$$u_x(0,t) = u_x(1,t) \ \ (0 \le t \le T), \tag{8}$$

$$(pt+q)^2 h''(t) - \alpha(pt+q)u_{txx}(x_0,t) - \beta u_{xx}(x_0,t) =$$
$$= a(t)h(t) + f(x_0,t) \ (0 \le t \le T). \tag{9}$$

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution of (1)-(5). By integrating both sides of Equation (1) with respect to x from 0 to 1, we find

$$(pt+q)^2 \frac{d^2}{dt^2}\int_0^1 u(x,t)dx - \alpha(pt+q)(u_{tx}(1,t) - u_{tx}(0,t)) - \beta(u_x(1,t) - u_x(0,t)) =$$

$$= a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \ \ (0 \le t \le T) \ . \tag{10}$$

Taking into account that $\int_0^1 f(x,t)dx = 0, \ (0 \le t \le T)$, allowing for (4),we have:

$$\alpha(pt+q)\frac{d}{dt}(u_x(1,t) - u_x(0,t)) + \beta(u_x(1,t) - u_x(0,t)) = 0 \ \ (0 \le t \le T) \ . \tag{11}$$

By (2) and $\varphi'(1) - \varphi'(0) = 0$ we get:

$$u_x(1,0) - u_x(0,0) = \varphi'(1) - \varphi'(0) = 0 \ . \tag{12}$$

Since the problem (11), (12) has only a trivial solution, then $u_x(1,t) - u_x(0,t) = 0 \ \ (0 \le t \le T)$, i.e. the condition (8) is fulfilled.

Now, from the equation (1) we find:

$$(pt+q)^2 \frac{d^2}{dt^2} u(x_0,t) - \alpha(pt+q)u_{txx}(x_0,t) - \beta u_{xx}(x_0,t) =$$
$$= a(t)u(x_0,t) + f(x_0,t) \ (0 \le t \le T). \tag{13}$$

Further, assuming $h(t) \in C^2[0,T]$ and twice differentiating (5), we have

$$\frac{d}{dt}u(x_0,t) = h'(t), \ \ \frac{d^2}{dt^2}u(x_0,t) = h''(t)\left(0 \le t \le T\right), \tag{14}$$

respectively.

From (13), by (5) and (14), we conclude that the relation (9) is fulfilled.

Now, suppose that $\{u(x,t), a(t)\}$ is the solution of (1)-(3), (8),(9). Then from (10), by means of (3) and (8), we find

$$(pt+q)^2 \frac{d^2}{dt^2}\int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx\left(0 \le t \le T\right). \tag{15}$$

By virtue of (2) and (6), it is not hard to see that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \quad \int_0^1 u_t(x,0)dx = \int_0^1 \psi(x)dx = 0. \qquad (16)$$

Since of problem (15), (16) has only a trivial solution, then $\int_0^1 u(x,t)dx = 0 \left(0 \le t \le T\right)$ i.e. condition (4) is fulfilled.

Further, from (9), (13),we obtain:

$$(pt+q)^2 \frac{d^2}{dt^2}(u(x_0,t)-h(t)) = a(t)(u(x_0,t)-h(t)) \ \ (0 \le t \le T) \qquad (17)$$

respectively.

Using (2) and the compatibility conditions (8) and (9), we have

$$,\ u(x_0,t)-h(0) = \varphi(x_0)-h(0) = 0, \ \ u_t(x_0,0)-h'(0) = \psi'(x_0)-h'(0) = 0 \quad . \ (18)$$

From (17), (18), we conclude that conditions (5) are satisfied. The theorem is thus proved.

3. Solvability of inverse boundary-value problem

Obviously, [9],

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \qquad (19)$$

is a basis in $L_2(0,1))$, where $\lambda_k = 2k\pi \ \ (k = 0,1,...)$. Since the system (26) forms basis in $L_2(0,1)$, it is obvious that for each solution $\{u(x,t), a(t)\}$ problems (1)-(3),(8),(9) first component $u(x,t)$ has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x \ \ (\lambda_k = 2\pi k)\ , \qquad (20)$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx \ \ , \ u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \ \ (k = 1,2,...),$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \ \ (k = 1,2,...).$$

Then, applying the formal scheme of the Fourier method, to determine the desired coefficients $u_{1k}(t) \ \ (k = 0,1,...)$, $u_{2k}(t) \ \ (k = 1,2,...)$ functions $u(x,t)$, from (1) and (2) we obtained:

$$(pt+q)^2 u''_{10}(t) = F_{10}(t;u,a) \ \ (0 \le t \le T) \ \ , \qquad (21)$$

$$(pt+q)^2 u''_{ik}(t) + \alpha(pt+q)\lambda_k^2 u'_{ik}(t) + \beta\lambda_k^2 u_{ik}(t) =$$
$$= F_{ik}(t;u,a) \ \ (i = 1,2; k = 1,2,...;\ 0 \le t \le T), \qquad (22)$$

$$u_{10}(0) = \varphi_{10},\ u'_{10}(0) = \psi_{10} \ \ , \qquad (23)$$

$$u_{ik}(0) = \varphi_{ik},\ u'_{ik}(0) = \psi_{ik} \ \ (i = 1,2; k = 1,...),, \qquad (24)$$

where

$$F_{1k}(t;u,a) = a(t)u_{1k}(t) + f_{1k}(t), \quad (k = 0,1,...) ,$$

$$f_{10}(t) = \int_0^1 f(x,t)dx \quad ,$$

$$f_{1k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx \;\; (k = 1,2,...) ,$$

$$\varphi_{10} = \int_0^1 \varphi(x)dx, \qquad \psi_{10} = 2\int_0^1 \psi(x)dx \;\; ,$$

$$\varphi_{1k} = 2\int_0^1 \varphi(x)\cos\lambda_k x dx, \psi_{1k} = 2\int_0^1 \psi(x)\cos\lambda_k x dx \qquad (k = 0,1,...),$$

$$F_{2k}(t) = a(t)u_{2k}(t) + f_{2k}(t),$$

$$f_{2k}(t) = 2\int_0^1 f(x,t)\sin\lambda_k x dx \quad (k = 1,2,...) ,$$

$$\varphi_{2k} = 2\int_0^1 \varphi(x)\sin\lambda_k x dx \quad (k = 1,2,...), \;\; \psi_{2k} = 2\int_0^1 \psi(x)\sin\lambda_k x dx \quad (k = 1,2,...).$$

Now, suppose that

$$\left(\frac{4\alpha\pi^2}{p} - 1\right)^2 - \frac{4\beta\pi^2}{p} > 0 \; .$$

Solving the problem (21)-(24) gives

$$u_{10}(t) = \phi_{10} + t\;\psi_{10} + \int_0^t \frac{(t-\tau)F_{10}(\tau;u,a)}{(p\tau+q)^2} d\tau \;\; (0 \le t \le T) \;\; , \tag{25}$$

$$u_{ik}(t) = \frac{1}{\gamma_k}\left[\left(\mu_{2k}\left(\frac{p}{q}t+1\right)^{\mu_{1k}} - \mu_{1k}\left(\frac{p}{q}t+1\right)^{\mu_{2k}}\right)\varphi_{ik} + \right.$$

$$+\frac{q}{p}\left(\left(\frac{p}{q}t+1\right)^{\mu_{2k}} - \left(\frac{p}{q}t+1\right)^{\mu_{1k}}\right)\psi_{ik}$$

$$\left. +\frac{1}{p}\int_0^t \frac{F_{ik}(\tau;u,a)}{(p\tau+q)^2}\left(\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}} - \left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}}\right)d\tau\right] (i = 1,2) \; , \tag{26}$$

where

$$\mu_{1k} = -\frac{1}{2}\left(\frac{\alpha\lambda_k^2}{p} - 1\right) - \sqrt{\frac{1}{4}\left(\frac{\alpha\lambda_k^2}{p} - 1\right)^2 - \frac{\beta\lambda_k^2}{p}} < 0,$$

$$\mu_{2k} = -\frac{1}{2}\left(\frac{\alpha\lambda_k^2}{p} - 1\right) + \sqrt{\frac{1}{4}\left(\frac{\alpha\lambda_k^2}{p} - 1\right)^2 - \frac{\beta\lambda_k^2}{p}} < 0$$

$$\gamma_k = \mu_{2k} - \mu_{1k} = 2\sqrt{\frac{1}{4}\left(\frac{\alpha\lambda_k^2}{p} - 1\right)^2 - \frac{\beta\lambda_k^2}{p}} \; .$$

To determine the first component of the classical solution to the problem (1)-(3), (8),(9) we substitute the expressions $u_{10}(t)$ $(k = 0,1,...)$, $u_{ik}(t)$ $(i = 1,2; k = 1,2,...)$ into (20) and obtain

$$u(x,t) = \varphi_{10} + t\,\psi_{10} + \int_0^t \frac{(t-\tau)F_{10}(\tau;u,a)}{(p\tau+q)^2}\,d\tau +$$
$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}\left(\frac{p}{q}t+1\right)^{\mu_{1k}} - \mu_{1k}\left(\frac{p}{q}t+1\right)^{\mu_{2k}}\right)\varphi_{1k} + \right.\right.$$
$$+\frac{q}{p}\left(\left(\frac{p}{q}t+1\right)^{\mu_{2k}} - \left(\frac{p}{q}t+1\right)^{\mu_{1k}}\right)\psi_{1k}$$
$$\left.\left.+\frac{1}{p}\int_0^t \frac{F_{1k}(\tau;u,a)}{(p\tau+q)^2}\left(\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}} - \left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}}\right)d\tau\right]\right\}\cos\lambda_k x +$$
$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}\left(\frac{p}{q}t+1\right)^{\mu_{1k}} - \mu_{1k}\left(\frac{p}{q}t+1\right)^{\mu_{2k}}\right)\varphi_{2k} + \right.\right.$$
$$+\frac{q}{p}\left(\left(\frac{p}{q}t+1\right)^{\mu_{2k}} - \left(\frac{p}{q}t+1\right)^{\mu_{1k}}\right)\psi_{2k}$$
$$\left.\left.+\frac{1}{p}\int_0^t \frac{F_{2k}(\tau;u,a)}{(p\tau+q)^2}\left(\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}} - \left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}}\right)d\tau\right]\right\}\sin\lambda_k x \, . \qquad (27)$$

Now, using (20), from (9) we find

$$a(t)=\left[h(t)\right]^{-1}\Big\{(pt+q)^2h''(t)-f\left(x_0,t\right)$$

$$+\sum_{k=1}^{\infty}\cos\lambda_k x_0\left(\alpha(pt+q)\lambda_{2k}^2u_{1k}'\left(t\right)+\beta\lambda_k^2u_{1k}\left(t\right)\right)+$$

$$+\sum_{k=1}^{\infty}\sin\lambda_k x_0\left(\alpha(pt+q)\lambda_{2k}^2u_k'\left(t\right)+\beta\lambda_k^2u_{2k}\left(t\right)\right)\Big\},\qquad(28)$$

Differentiating (26) two times, we get:

$$u_{ik}'(t)=\frac{1}{\gamma_k}\Bigg[\mu_{1k}\mu_{2k}\frac{p}{q}\left(\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}-\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}\right)\varphi_{ik}+$$

$$+\left(\mu_{2k}\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}-\mu_{1k}\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}\right)\psi_{ik}+$$

$$+\int_0^t\frac{F_{ik}(\tau;u,a)}{(p\tau+q)^2}\left(\mu_{2k}\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}-1}-\mu_{1k}\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}-1}\right)d\tau\Bigg]\ (i=1,2),\qquad(29)$$

$$u_{ik}''(t)=\frac{1}{\gamma_k}\Bigg[\mu_{1k}\mu_{2k}\left(\frac{p}{q}\right)^2\left((\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-2}-(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-2}\right)\varphi_{ik}+$$

$$+\frac{p}{q}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}-\mu_{1k}(\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}\right)\psi_{ik}+$$

$$+p\int_0^t\frac{F_{ik}(\tau;u,a)}{(p\tau+q)^3}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}-2}-\mu_{1k}(\mu_{1k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}-2}\right)d\tau\Bigg]+$$

$$+\frac{F_{ik}(t;u,a)}{(pt+q)^2}\ (i=1,2)\ .\qquad(30)$$

By virtue of (22) and (30) we have:

$$\alpha(pt+q)\lambda_k^2u_{ik}'(t)+\beta\lambda_k^2u_{ik}(t)=F_{ik}(t;u,a)-(pt+q)^2u_{ik}''(t)=$$

$$=-\frac{(pt+q)^2}{\gamma_k}\Bigg[\mu_{1k}\mu_{2k}\left(\frac{p}{q}\right)^2\left((\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-2}-(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-2}\right)\varphi_{ik}+$$

$$+\frac{p}{q}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}-\mu_{1k}(\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}\right)\psi_{ik}$$

$$+p\int_0^t \frac{F_{ik}(\tau;u,a)}{(p\tau+q)^3}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}-2}-\right.$$

$$\left.\left.-\mu_{1k}(\mu_{1k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}-2}\right)d\tau\right]\ (i=1,2;k=1,2,...)\ . \qquad (31)$$

Putting the expression of (31) in (28) we obtain

$$a(t)=\left[h(t)\right]^{-1}\left\{(pt+q)^2h''(t)-f\left(x_0,t\right)-\right.$$

$$-\sum_{k=1}^{\infty}\cos\lambda_k x_0\,\frac{(pt+q)^2}{\gamma_k}\ \times$$

$$\times\left[\ \mu_{1k}\mu_{2k}\left(\frac{p}{q}\right)^2\left((\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-2}-(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-2}\right)\varphi_{1k}\ +\right.$$

$$+\frac{p}{q}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}-\mu_{1k}(\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}\right)\psi_{1k}$$

$$+p\int_0^t \frac{F_{1k}(\tau;u,a)}{(p\tau+q)^3}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}-2}-\right.$$

$$\left.\left.-\mu_{1k}(\mu_{1k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}-2}\right)d\tau\right]-$$

$$-\sum_{k=1}^{\infty}\sin\lambda_k x_0\,\frac{(pt+q)^2}{\gamma_k}\ \times$$

$$\times\left[\ \mu_{1k}\mu_{2k}\left(\frac{p}{q}\right)^2\left((\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-2}-(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-2}\right)\varphi_{2k}\ +\right.$$

$$+\frac{p}{q}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{2k}-1}-\mu_{1k}(\mu_{1k}-1)\left(\frac{p}{q}t+1\right)^{\mu_{1k}-1}\right)\psi_{2k}$$

$$+\,p\int_0^t \frac{F_{2k}(\tau;u,a)}{(p\tau+q)^3}\left(\mu_{2k}(\mu_{2k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{2k}-2}-\right.$$

$$\left.\left.\left.-\,\mu_{1k}(\mu_{1k}-1)\left(\frac{pt+q}{p\tau+q}\right)^{\mu_{1k}-2}\right)d\tau\right]\right\}, \quad (32)$$

Thus, the solution of problem (1)–(3), (8),(9) was reduced to the solution of system (27), (32) with respect to unknown functions $u(x,t)$ and $a(t)$.

The following lemma is valid.

Lemma 1. *If* $\{u(x,t),a(t)\}$ *is any solution to problem (1) - (3), (8),(9), then the functions*

$$u_{10}(t)=\int_0^1 u(x,t)dx\ ,\ u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx\ \ (k=1,2,...),$$

$$u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx\ \ (k=1,2,...).$$

satisfies the system (25), (26) in $C[0,T]$.

It follows from Lemma 1 that

Corollary 1. *Let system (27), (32) have a unique solution. Then problem (1) - (3), (8), (9) cannot have more than one solution, i.e. if the problem (1) - (3), (8), (9) has a solution, then it is unique.*

With the purpose to study the problem (1) - (3), (8), (9), we consider the following functional spaces.

Denote by $B_{2,T}^3$ [9] a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x \quad (\lambda=2k\pi),$$

defined on D_T such that the functions $u_{1k}(t)$ $(k=0,1,2,...)$, $u_{2k}(t)$ $(k=1,2,...)$ are continuous on $[0,T]$ and

$$\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{1k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<+\infty.$$

The norm on this set is given by

$$\|u(x,t)\|_{B_{2,T}^3}=\|u_{10}(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{1k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.$$

It is known that $B_{2,T}^3$ is Banach space .

Obviously, $E_T^3 = B_{2,T}^3 \times C[0,T]$ is also Banach space, where the norm of an element $z = \{u, a, b\}$ is determined by the formula

$$\|z(x,t)\|_{E_T^3} = \|u(x,t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]}.$$

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

in the space E_T^3, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)\sin\lambda_k x,$$

$$\Phi_2(u,a) = \tilde{a}(t)$$

and the functions $\tilde{u}_{10}(t)$, $\tilde{u}_{ik}(t)$, $(i = 1,2; k = 1,2,...,)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (25), (26),(32) respectively.

It is easy to see that

$$\mu_{ik} < 0, \left(\frac{p}{q}t + 1\right)^{\mu_{ik}} < 1, \left(\frac{pt+q}{p\tau+q}\right)^{\mu_{ik}-j} < 1 \ ,$$

$$(i = 1,2;\ j = 0,1,2; k = 1,2,...;\ 0 \le t \le T;\ 0 \le \tau \le t),$$

$$|\mu_{ik}| \le \frac{1}{2}\left(\frac{\alpha\lambda_k^2}{p} - 1\right) + \sqrt{\frac{1}{4}\left(\frac{\alpha\lambda_k^2}{p} - 1\right)^2 - \frac{\beta\lambda_k^2}{p}} \le \frac{\alpha}{p}\lambda_k^2 \ (i = 1,2; k = 1,2,...),$$

$$|\mu_{1k}\mu_{2k}| = \frac{\beta}{p}\lambda_k^2 \ , \frac{1}{\gamma_k} = \frac{1}{2\sqrt{\frac{1}{4}\left(\frac{\alpha\lambda_k^2}{p} - 1\right)^2 - \frac{\beta\lambda_k^2}{p}}} \le \frac{p}{\alpha\lambda_k^2} \equiv \frac{\gamma_0}{\lambda_k^2}.$$

Given these relationships, we have:

$$\|\tilde{u}_{10}(t)\|_{C[0,T]} \le |\varphi_0| + T|\psi_0| + \frac{T\sqrt{T}}{q^2}\left(\int_0^T |f_{10}(\tau)|^2 d\tau\right)^{\frac{1}{2}} + $$

$$+ \frac{T^2}{q^2}\|a(t)\|_{C[0,T]}\|u_{10}(t)\|_{C[0,T]} \quad , \tag{33}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \frac{2\sqrt{5}\alpha\gamma_0}{p}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_{ik}|)^2\right)^{\frac{1}{2}} + \frac{2\sqrt{5}q\gamma_0}{p}\left(\sum_{k=1}^{\infty}(\lambda_k^2|\psi_{ik}|)^2\right)^{\frac{1}{2}} +$$

$$+\frac{2\gamma_0}{pq^2}\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\frac{2\sqrt{5}\gamma_0}{pq^2}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$(i=1,2), \tag{34}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|(pt+q)^2h''(t)-f\left(x_0,t\right)\right\|_{C[0,T]}+$$

$$+4(pT+q)^2\gamma_0\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sum_{i=1}^{2}\left[\frac{\alpha\beta}{q^2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\phi_{ik}|)^2\right)^{\frac{1}{2}}+\right.$$

$$+\frac{\alpha^2}{pq^3}\left(\sum_{k=1}^{\infty}(\lambda_k^2|\psi_{ik}|)^2\right)^{\frac{1}{2}}+\frac{\alpha^2}{pq^3}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.+\frac{\alpha^2}{pq^3}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\Bigg\}$$

Let us assume that the data of problem (1)-(3), (8)-(9) satisfy the following conditions:

1. $\alpha>0,\ \beta>0,\ \frac{\alpha^2}{8}-\beta>0.$
2. $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=\varphi(1),\ \varphi'(0)=\varphi'(1),\varphi''(0)=\varphi''(1).$
3. $\psi(x)\in C^2[0,1],\ \psi'''(x)\in L_2(0,1)$,
$\psi(0)=\psi(1),\ \psi'(0)=\psi'(1),\ \psi''(0)=\psi''(1).$
4. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t)\in C(D_T),\ f_{xxx}(x,t)\in L_2(D_T),$
$f(0,t)=f(1,t),\ f_x(0,t)=f_x(1,t),\ f_{xx}(0,t)=f_{xx}(1,t)\ (0\leq t\leq T).$
5. $h(t)\in C^2[0,T]$, $h(t)\neq 0\ \ (0\leq t\leq T)$.

Then from (33)-(35) we get:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3}\leq A_1(T)+B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,, \tag{36}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\leq A_2(T)+B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^3}\,, \tag{37}$$

where

$$A_1(T)=\|\varphi(x)\|_{L_2(0,1)}+T\|\psi(x)\|_{L_2(0,1)}+\frac{T\sqrt{T}}{q^2}\|f(x,t)\|_{L_2(D_T)}+$$

$$+\frac{2\sqrt{5}\alpha\gamma_0}{p}\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\frac{2\sqrt{5}\alpha\gamma_0}{p}\left\|\psi''(x)\right\|_{L_2(0,1)}+\frac{2\gamma_0}{pq^2}\sqrt{5T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\frac{T^2}{q^2}+\frac{2\sqrt{5}\gamma_0}{pq^2}T,$$

$$A_2(T)=\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\left\{\left\|(pt+q)^2h''(t)-f\left(x_0,t\right)\right\|_{C[0,T]}+\right.$$

$$+8(pT+q)^2\gamma_0\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\frac{\alpha\beta}{q^2}{}_0\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\right.$$

$$\left.\left.+\frac{\alpha^2}{pq^3}\left\|\psi''(x)\right\|_{L_2(0,1)}+\frac{\alpha^2}{pq^3}\sqrt{T}\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T)=4\left\|\left[h(t)\right]^{-1}\right\|_{C[0,T]}\frac{\alpha^2\gamma_0(pT+q)^2}{pq^3}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T,$$

From inequalities (36),(37) we conclude

$$\left\|\tilde{u}(x,t)\right\|_{B^3_{2,T}}+\left\|\tilde{a}(t)\right\|_{C[0,T]}\le A(T)+B(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}, \qquad (38)$$

where

$$A(T)=A_1(T)+A_2(T),\ \ B(T)=B_1(T)+B_2(T),$$

Thus, we can prove the following theorem

Theorem 2. *Assume that statements 1-5 and the condition*

$$(B(T)A(T)+2)^2<1 \qquad (39)$$

holds, then problem (1)-(3), (8),(9) has a unique solution in the ball $K=K_R(\|z\|_{E^3_T}\le R\le A(T)+2)$ *of the space* E^3_T.

Proof. In the space E^3_T , consider the operator equation

$$z=\Phi z, \qquad (40)$$

where $z=\{u,a\}$, and the components $\Phi_i(u,a)$ $(i=1,2)$, of operator $\Phi(u,a)$ defined by the right sides of (27) and (32) .

Consider the operator $\Phi(u,a)$ in the ball $K=K_R$ out of E^3_T .Similarly to (38), we obtain that for any the estimates are valid: respectively and the following inequalities hold:

$$\|\Phi z\|_{E_T^3} \le A(T)+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3} \le A(T)+B(T)\left(A(T)+2\right)^2, \qquad (41)$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^3} \le 2B(T)R\left(\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^{3,3}}+\|a_1(t)-a_2(t)\|_{C[0,T]}\right) \quad . \qquad (42)$$

Then it follows from (39), (41), and (42) that the operator Φ acts in the ball $K=K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\}=\{u,a\}$ in the ball $K=K_R$, which is a solution of equation (40); i.e. the pair $\{u,a,b\}$ is the unique solution of the systems (27) and (32) in $K=K_R$.

Hen the function $u(x,t)$ as an element of space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t)$ in $\overline{D}_T$.

Now, from (29) we get:

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u'_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \frac{2\sqrt{5}\beta\gamma_0}{q}\|\varphi'''(x)\|_{L_2(0,1)}+\frac{2\sqrt{5}\alpha\gamma_0}{p}\|\psi'''(x)\|_{L_2(0,1)}+$$

$$+\frac{2\sqrt{5}\alpha\gamma_{0\sqrt{T}}}{pq^2}\|f_{xxx}(x,t)\|_{L_2(D_T)}+\frac{2\sqrt{5}\alpha\gamma_{0T}}{pq^2}\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}.$$

This implies that $u_t(x,t),\ u_{tx}(x,t), u_{txx}(x,t)$ are continuous in $\overline{D}_T$.

Further, from (22) we have:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\|u''_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \frac{2\alpha}{q}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u'_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\frac{2\alpha}{q^2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\frac{2}{q^2}\left\|\|f_x(x,t)+a(t)u_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}(i=1,2).$$

It is clear from the last relation that $u_{tt}(x,t)$ is continuous in $\overline{D}_T$.

It is easy to verify that Eq. (1) and conditions (2), (3), (8), (9) satisfy in the usual sense. So, $\{u(x,t),a(t)\}$ is a solution of (1)-(3), (8), (9), and by Lemma 1 it is unique in the ball $K=K_R$. The proof is complete.

In summary, from Theorem 1 and Theorem 2, straightforward implies the unique solvability of the original problem (1) - (5).

Theorem 3. Suppose that all assumptions of Theorem 2, $\int_0^1 f(x,t)dx = 0 \qquad (0 \le t \le T)$ and the compatibility conditions (6),(7) holds. Then problem (1) - (5) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^3} \le A(T)+2)$ of the space E_T^3 .

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