MEROMORPHIC SUBCLASSES OF P-VALENT FUNCTIONS INVOLVING CERTAIN OPERATOR

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Abstract: In this paper we investigate some inclusion relationships of two new subclassses of meromorphically p-valent functions, defined by means of a linear operator. We also study some integral preserving properties and convolution properties of these classes.

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1. Introduction

For any integer $m > -p$, let $\sum_{p,m}$ denote the class of all meromorphic functions by:

$$f(z) = z^{-p} + \sum_{n=m}^{\infty} a_n z^n \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and -valent in a punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. For convenience, we write $\sum_{p,-p+1} = \sum_p$.
The class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \;, z \in \mathbb{U},$$

is denoted by $\mathbb{A}$. The functions of this class is called starlike of order $\alpha, 0 \le \alpha < 1$ if

$$R \frac{zf'(z)}{f(z)} > \alpha$$

and called prestarlike of order $\alpha, 0 \le \alpha < 1$ if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha),$$

where we denote by $S^*(\alpha)$ and $R(\alpha)$ the classes of starlike and prestarlike of order

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function , which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z \in \mathrm{U}$, such that $f(z)=g(w(z)),\ z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [2], [5] and [6]):

$$f(z) \prec g(z) \Leftrightarrow f(0)=g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

For functions $f(z) \in \sum_{p,m}$ given by (1.1) and $g(z) \in \sum_{p,m}$ given by $g(z)=z^{-p}+\sum_{n=m}^{\infty} b_n z^n$, the Hadamard product of $f(z)$ and $g(z)$ is given by:

$$(f*g)(z)=z^{-p}+\sum_{n=m}^{\infty} a_n b_n z^n=(g*f)(z). \tag{2}$$

For complex numbers $\alpha_1,\alpha_2,...,\alpha_l$ and $\beta_1,\beta_2,...,\beta_s$ ($\beta_j \notin Z_0^- =\{0,-1,-2,...\}$; $j=1,2,...,s$), the generalized hypergeometric function ${}_lF_s(\alpha_1,...,\alpha_l;$ $\beta_1,...,\beta_s;z)$ (see, for example, [11]) is given by:

$${}_lF_s(\alpha_1,...,\alpha_l;\beta_1,...,\beta_s;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_s)_n(1)n}z^n$$

$$l(l \le s+1; s,l \in \mathrm{N}_0=\mathrm{N}\cup\{0\}; z \in \mathrm{U}), \tag{3}$$

where

$$(d)_k=\begin{cases}1 & (k=0; d \in \mathrm{C}^*=\mathrm{C}\setminus\{0\})\\ d(d+1)...(d+k-1) & (k \in \mathrm{N}; d \in \mathrm{C}).\end{cases}$$

Using the function $\Omega_{p,l,s}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_s;z):\ \sum_{p,m}\to\sum_{p,m}$:

$$\Omega_{p,l,s}(\alpha_1)=\Omega_{p,l,s}(\alpha_1,...,\alpha_l;\beta_1,...,\beta_s;z)=z^{-p}\ {}_lF_s(\alpha_1,...,\alpha_l;\beta_1,...,\beta_s;z)$$
$$=z^{-p}+\sum_{n=1-p}^{\infty}\frac{(\alpha_1)_{n+p}...(\alpha_l)_{n+p}}{(\beta_1)_{n+p}...(\beta_s)_{n+p}(1)_{n+p}}z^n, \tag{4}$$

For $f \in \sum_{p,m}$, Mostafa [8] defined a function $\Omega^*_{p,l,s}(\alpha_1)$ by:

$$\Omega_{p,l,s}(\alpha_1)*\Omega^*_{p,l,s}(\alpha_1)=\frac{1}{z^p(1-z)^{\lambda+p}}\quad (z \in \mathrm{U}^*; \lambda>-p), \tag{5}$$

and defined the family of linear operators $M^{\lambda}_{p,l,s}(\alpha_1):\ \sum_{p,m}\to\sum_{p,m}$ given by:

$$M^{\lambda}_{p,l,s}(\alpha_1) = \Omega^{*}_{p,l,s}(\alpha_1) * f(z)$$
$$= z^{-p} + \sum_{n=m}^{\infty} \frac{(\beta_1)_{n+p}\cdots(\beta_s)_{n+p}(\lambda+p)_{n+p}}{(\alpha_1)_{n+p}\cdots(\alpha_l)_{n+p}} a_n z^n \; (\lambda > -p; \alpha_i > 0). \tag{6}$$

From equation (6) , it can be easily verify that:

$$z(M^{\lambda}_{p,l,s}(\alpha_1+1)f(z))' = \alpha_1 M^{\lambda}_{p,l,s}(\alpha_1)f(z) - (\alpha_1+p)M^{\lambda}_{p,l,s}(\alpha_1+1)f(z) \tag{7}$$

and

$$z(M^{\lambda}_{p,l,s}(\alpha_1)f(z))' = (\lambda+p)M^{\lambda+1}_{p,l,s}(\alpha_1)f(z) - (\lambda+2p)M^{\lambda}_{p,l,s}(\alpha_1)f(z). \tag{8}$$

For $\alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c, a, c > 0$ we have $M^{\lambda}_{p,2,1}(\alpha_1)f(z) = L^{\lambda}_{p}(a,c)f(z)$ introduced and studied by Aouf et al. [1]. Also we have

$i)$ $M^{0}_{p,2,1}(p,p;p)f(z) = M^{1}_{p,2,1}(p+1,p;p)f(z) = f(z);$

$ii)$ $M^{1}_{p,2,1}(p,p;p)f(z) = \dfrac{2pf(z)+zf'(z)}{p};$

$iii)$ $M^{2}_{p,2,1}(p+1,p;p)f(z) = \dfrac{(2p+1)f(z)+zf'(z)}{p+1};$

For more specializations of the parameters $\lambda, \alpha_i \; (i=1,2,...l), \beta_j \; (j=1,2,...,s), l$ and , in (6), (see [8]) .

Let P be the class of functions $h(z)$ with $h(0)=1,$ Re $h(z)>0$ which are convex univalent in U .

For $p,k \in N, \alpha_i, \beta_j \notin Z_0^-$ be real, $\in_k = e^{2\pi/k}$, let

$$f^{\lambda}_{k}(\alpha_1)(z) = \frac{1}{k}\sum_{j=0}^{k-1} \in_k^{jp} M^{\lambda}_{p,l,s}(\alpha_1) f(\in_k^{j} z) = z^{-p} + ..., \; f \in \sum\nolimits_{p,m}. \tag{9}$$

By (7) and (8) $f^{\lambda}_{k}(\alpha_1,\beta_1)(z)$ satisfies:

$$z(f^{\lambda}_{k}(\alpha_1+1)(z))' = \alpha_1 f^{\lambda}_{k}(\alpha_1)(z) - (\alpha_1+p) f^{\lambda}_{k}(\alpha_1+1)(z) \tag{10}$$

and

$$z(f^{\lambda}_{k}(\alpha_1)(z))' = (\lambda+p) f^{\lambda+1}_{k}(\alpha_1)(z) - (\lambda+2p) f^{\lambda}_{k}(\alpha_1)(z). \tag{11}$$

Definition **.** *For* $h \in P$, $f \in \sum_{p,m},$ $f^{\lambda}_{k}(\alpha_1)(z) \neq 0, z \in U^{*},$ $S^{\lambda}_{k}(\alpha_1,\beta_1,h)$ *is the class of functions* f *satisfying:*

$$-\frac{z(M_{p,l,s}^{\lambda}(\alpha_1)f(z))'}{pf_k^{\lambda}(\alpha_1)(z)} \prec h(z) \tag{12}$$

and $K_k^{\lambda}(\alpha_1,\beta_1,h)$ *is the class of functions* f *satisfying:*

$$-\frac{z(M_{p,l,s}^{\lambda}(\alpha_1)f(z))'}{pg_k^{\lambda}(\alpha_1)(z)} \prec h(z), \tag{13}$$

where $g_k^{\lambda}(\alpha_1,\beta_1)(z)\neq 0,$ *is defined as in (9)*

To prove our results, we need the following lemmas.

Lemma ***[3].*** *Let* $\beta,\gamma\in \mathbb{C}, \beta\neq 0, h$ *be convex univalent with* $\operatorname{Re}\{\beta h(z)+\gamma\}>0$ *and* q *be an analytic function such that* $q(0)=h(0).$ *If*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z),$$

then

$$q(z)\prec h(z).$$

Lemma ***[7].*** *Let* h *be convex univalent and* w *be analytic,* $\operatorname{Re} w\geq 0.$ *If the analytic function* q *satisfies* $q(0)=h(0)$ *and*

$$q(z)+w(z)zq'(z)\prec h(z),$$

then $q(z)\prec h(z).$

Lemma ***[10].*** *For* $\alpha<1, f\in R(\alpha)$ *and* $\varphi\in S^*(\alpha),$ *we have for any analytic function* F *in* $\mathbb{U},$

$$\frac{f*(\varphi F)}{f*\varphi}(\mathbb{U})\subset \overline{co}(F(\mathbb{U}),$$

where $\overline{co}(F(\mathbb{U})$ *is the convex hull of* $(F(\mathbb{U}).$

2- Main Results

Theorem 1 . *If* $f\in S_k^{\lambda}(\alpha_1,\beta_1,h),$ *then*

$$-\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)} \prec h(z), \tag{14}$$

where $f_k^{\lambda}(\alpha_1)(z)$ *is defined as in (9).*

Proof. From (9) we have:

$$f_k^{\lambda}(\alpha_1)(\in_k^j z) = \frac{1}{k}\sum_{t=0}^{k-1} \in_k^{jt} M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j+t} z)$$

$$= \frac{\in_k^{-jp}}{k}\sum_{t=0}^{k-1} \in_k^{(j+t)p} M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j+t} z) \tag{15}$$

$$= \in_k^{-jp} f_k^{\lambda}(\alpha_1)(z)$$

and

$$\left(f_k^{\lambda}(\alpha_1)(z)\right)' = \frac{1}{k}\sum_{j=0}^{k-1} \in_k^{j(p+1)} \left(M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j+t} z)\right)'. \tag{16}$$

By (15) and (16), we have

$$-\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)} = -\frac{1}{k}\sum_{j=0}^{k-1} \frac{\in_k^{j(p+1)}\left(M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j} z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}$$

$$= -\frac{1}{k}\sum_{j=0}^{k-1} \frac{\in_k^{j}\left(M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j} z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}. \tag{17}$$

Since $f \in S_k^{\lambda}(\alpha_1,\beta_1,h)$, we have,

$$-\frac{\in_k^{j}\left(M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^{j} z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)} \prec h(z),$$

which leads to (14).

Theorem 2. *For* $\alpha_1 > 0, h \in \mathrm{P}$ *with* $Rh(z) < 1 + \frac{\alpha_1}{p}$ *and for* $f \in S_k^{\lambda}(\alpha_1,\beta_1,h)$, $f_k^{\lambda}(\alpha_1+1)(z) \neq 0$, *we have,* $f \in S_k^{\lambda}(\alpha_1+1,\beta_1,h)$.

Proof. Since $f \in S_k^{\lambda}(\alpha_1,\beta_1,h)$, then the function

$$q(z) = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1+1) f(z)\right)'}{pf_k^{\lambda}(\alpha_1+1)(z)}, \tag{18}$$

is analytic and $q(0)=1$. Applying (7) in (18) we have

$$q(z) f_k^{\lambda}(\alpha_1+1)(z) = -\frac{1}{p}[\alpha_1 M_{p,l,s}^{\lambda}(\alpha_1) f(z) - (p+\alpha_1) M_{p,l,s}^{\lambda}(\alpha_1+1) f(z)]. \tag{19}$$

Differentiating (19) and using (7) again, we have

$$\left(\alpha_1 + p + \frac{z\left(f_k^{\lambda}(\alpha_1+1)(z)\right)'}{f_k^{\lambda}(\alpha_1+1)(z)}\right) q(z) + zq'(z) = -\frac{\alpha_1 z\left(M_{p,l,s}^{\lambda}(\alpha_1) f(z)\right)'}{pf_k^{\lambda}(\alpha_1+1)(z)}. \tag{20}$$

Taking

$$\phi(z) = -\frac{z\left(f_k^{\lambda}(\alpha_1+1)(z)\right)'}{pf_k^{\lambda}(\alpha_1+1)(z)}, \tag{21}$$

we see that $\phi(z)$ is analytic, $\phi(0)=1$ and (20) can be written as

$$\left(\alpha_1 + p - p\phi(z)\right)q(z) + zq'(z) = -\frac{\alpha_1 z\left(M^{\lambda}_{p,l,s}(\alpha_1) f(z)\right)'}{pf_k^{\lambda}(\alpha_1+1)(z)}, \tag{22}$$

that is

$$q(z) + \frac{zq'(z)}{\alpha_1 + p - p\phi(z)} = -\frac{\alpha_1 z\left(M^{\lambda}_{p,l,s}(\alpha_1) f(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}. \tag{23}$$

Since $f \in S_k^{\lambda}(\alpha_1, \beta_1, h)$, (23) implies

$$q(z) + \frac{zq'(z)}{\alpha_1 + p - p\phi(z)} \prec h(z). \tag{24}$$

Combining (10) and (21) , we have

$$\alpha_1 + p - p\phi(z) = \frac{\alpha_1 f_k^{\lambda}(\alpha_1)(z)}{pf_k^{\lambda}(\alpha_1+1)(z)}. \tag{25}$$

Differentiating (25) we get

$$\phi(z) + \frac{z\phi'(z)}{\alpha_1 + p - p\phi(z)} = -\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}. \tag{26}$$

By Theorem 1, we have

$$-\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)} \prec h(z),$$

which yields

$$\phi(z) + \frac{z\phi'(z)}{\alpha_1 + p - p\phi(z)} \prec h(z).$$

Since $R\{\alpha_1 + p - ph(z)\} > 0$, by Lemma 1, we have $\phi(z) \prec h(z)$, which implies $R\{\alpha_1 + p - p\phi(z)\} > 0$. Applying Lemma 2 and from (25) we have $q(z) \prec h(z)$ that is $f \in S_k^{\lambda}(\alpha_1+1, \beta_1, h)$.

Theorem 3. *Let* $\alpha_1 > 0, h \in \mathrm{P}$ *with* $R\{p + \alpha_1 - ph(z)\} > 0$ *and* $f \in K_k^{\lambda}(\alpha_1, \beta_1, h)$ *with* $g \in S_k^{\lambda}(\alpha_1, \beta_1, h)$. *Then,* $f \in K_k^{\lambda}(\alpha_1+1, \beta_1, h)$ *provided* $g_k^{\lambda}(\alpha_1)(z) \neq 0$.

Proof. By Theorem 2, $g \in S_k^{\lambda}(\alpha_1, \beta_1, h) \Rightarrow g \in S_k^{\lambda}(\alpha_1+1, \beta_1, h)$ and by Theorem 1, we have

$$\psi(z) = -\frac{z\left(g_k^{\lambda}(\alpha_1+1)(z)\right)'}{pg_k^{\lambda}(\alpha_1+1)(z)} \prec h(z). \tag{27}$$

Let

$$q(z) = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1+1)f(z)\right)'}{pg_k^{\lambda}(\alpha_1+1)(z)}. \quad (28)$$

Then, from (7) , we have

$$q(z)g_k^{\lambda}(\alpha_1+1)(z) = -\frac{1}{p}[\alpha_1 M_{p,l,s}^{\lambda}(\alpha_1)f(z) - (p+\alpha_1)M_{p,l,s}^{\lambda}(\alpha_1+1)f(z)]. \quad (29)$$

Differentiating (29) we have

$$\left(\alpha_1 + p - p\psi(z)\right)q(z) + zq'(z) = -\frac{\alpha_1 z\left(M_{p,l,s}^{\lambda}(\alpha_1)f(z)\right)'}{pg_k^{\lambda}(\alpha_1+1)(z)}. \quad (30)$$

Applying (10) for g , (30) is equivalent to

$$q(z) + \frac{zq'(z)}{\alpha_1 + p - p\psi(z)} = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1)f(z)\right)'}{pg_k^{\lambda}(\alpha_1)(z)}. \quad (31)$$

Since $f \in K_k^{\lambda}(\alpha_1, \beta_1, h)$, the above equation leads to

$$q(z) + \frac{zq'(z)}{\alpha_1 + p - p\psi(z)} \prec h(z). \quad (32)$$

We have $R\{p+\alpha_1 - p\psi(z)\} > 0$ because $R\{p+\alpha_1 - ph(z)\} > 0.$ Applying Lemma 2, for (2.19) , we have $q(z) \prec h(z).$

Theorem 4. *Let* $h \in \mathrm{P}, R\{2p+\lambda - ph(z)\} > 0$ *and* $f \in S_k^{\lambda+1}(\alpha_1, \beta_1, h)$ *such that* $f_k^{\lambda+1}(\alpha_1)(z) \neq 0.$ *Then* $f \in S_k^{\lambda}(\alpha_1, \beta_1, h).$

Proof . Let $f \in S_k^{\lambda+1}(\alpha_1, \beta_1, h),$

$$q(z) = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1)f(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}. \quad (33)$$

Applying (8) in (33) we have

$$q(z)f_k^{\lambda}(\alpha_1)(z) = -\frac{p+\lambda}{p}[M_{p,l,s}^{\lambda+1}(\alpha_1)f(z) + (\frac{2p+\lambda}{p})M_{p,l,s}^{\lambda}(\alpha_1)f(z)]. \quad (34)$$

Differentiating (34) and putting

$$\Phi(z) = -\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}, \quad (35)$$

simple computations leads to

$$\left[\lambda + 2p - p\Phi(z)\right]q(z) + zq'(z) = -\left(\frac{p+\lambda}{p}\right)\frac{z\left(M_{p,l,s}^{\lambda+1}(\alpha_1)f(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)}. \quad (36)$$

Using (11) we have

$$\lambda+2p-p\Phi(z)=\frac{(\lambda+p)f_k^{\lambda+1}(\alpha_1)(z)}{f_k^{\lambda}(\alpha_1)(z)}. \qquad (37)$$

So, (36) , reduces to

$$q(z)+\frac{zq'(z)}{\lambda+2p-p\Phi(z)}=-\frac{z\left(M_{p,l,s}^{\lambda+1}(\alpha_1)f(z)\right)'}{pf_k^{\lambda+1}(\alpha_1)(z)}\prec h(z), \qquad (38)$$

where $f\in S_k^{\lambda+1}(\alpha_1,\beta_1,h)$. Also differentiating (2.24) , we have

$$\Phi(z)+\frac{z\Phi'(z)}{\lambda+2p-p\Phi(z)}=-\frac{z\left(f_k^{\lambda+1}(\alpha_1)f(z)\right)'}{pf_k^{\lambda+1}(\alpha_1)(z)}. \qquad (39)$$

By Theorem 1, we have

$$-\frac{z\left(f_k^{\lambda+1}(\alpha_1)f(z)\right)'}{pf_k^{\lambda+1}(\alpha_1)(z)}\prec h(z). \qquad (40)$$

Combining (39) , (40) and the condition $R\{\lambda+2p-ph(z)\}>0$, we have $\Phi(z)\prec h(z)$, which leads to $R\{\lambda+2p-p\Phi(z)\}>0$ and so applying Lemma 2 to (38) , we have $q(z)\prec h(z)$ which complete the proof of Theorem 4.

Theorem 5. *Let* $h\in P$ *with* $R\{\lambda+2p-ph(z)\}>0$ *and* $f\in K_k^{\lambda+1}(\alpha_1,\beta_1,h)$ *with* $g\in S_k^{\lambda+1}(\alpha_1,\beta_1,h)$. *Then,* $f\in K_k^{\lambda}(\alpha_1,\beta_1,h)$ *provided* $g_k^{\lambda}(\alpha_1)(z)\neq 0$.

Proof. By Theorem 4, $g\in S_k^{\lambda+1}(\alpha_1,\beta_1,h)\Rightarrow g\in S_k^{\lambda}(\alpha_1,\beta_1,h)$ and by Theorem 1, we have

$$\Psi(z)=-\frac{z\left(g_k^{\lambda}(\alpha_1)f(z)\right)'}{pg_k^{\lambda}(\alpha_1)(z)}\prec h(z),$$

and letting

$$q(z)=-\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1)f(z)\right)'}{pg_k^{\lambda}(\alpha_1)(z)},$$

we can complete the proof as in Theorem 4.

Next, let

$$F_{p,\mu}(f(z))=\frac{\mu-p}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt \quad (\mu>0), \qquad (41)$$

which by using (6) gives

$$\mu M_{p,l,s}^{\lambda}(\alpha_1)F_{p,\mu}f(z)+z\left(M_{p,l,s}^{\lambda+1}(\alpha_1)F_{p,\mu}f(z)\right)'=(\mu-p)M_{p,l,s}^{\lambda}(\alpha_1)f(z). \qquad (42)$$

The operator $F_{p,\mu}$ was investigated by many authors (see [12] and [13]).

Theorem 6. Let $h \in \mathrm{P}$ with $R\{\mu - ph(z)\} > 0$ and $f \in S_k^{\lambda}(\alpha_1, \beta_1, h),$ then $F_{p,\mu}(f) \in S_k^{\lambda}(\alpha_1, \beta_1, h)$ provided $F_k^{\lambda}(\alpha_1)(z) \neq 0,$ where $F_k^{\lambda}(\alpha_1)(z)$ is defined as (9) .

Proof. From (42) we have

$$\mu F_k^{\lambda}(\alpha_1)(z) + z\left(F_k^{\lambda}(\alpha_1)(z)\right)' = (\mu - p) f_k^{\lambda}(\alpha_1)(z). \tag{43}$$

Let

$$q(z) = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1) F_{p,\mu}(f(z))\right)'}{pF_k^{\lambda}(\alpha_1)(z)}$$

and

$$w(z) = -\frac{z\left(F_k^{\lambda}(\alpha_1)(z)\right)'}{pF_k^{\lambda}(\alpha_1)(z)}. \tag{44}$$

Using (43) in (44) , we have

$$\mu - pw(z) = (\mu - p)\frac{f_k^{\lambda}(\alpha_1)(z)}{F_k^{\lambda}(\alpha_1)(z)}. \tag{45}$$

Differentiating (45) and using Theorem 1, we obtain

$$w(z) + \frac{zw'(z)}{\mu - pw(z)} = -\frac{z\left(f_k^{\lambda}(\alpha_1)(z)\right)'}{pf_k^{\lambda}(\alpha_1)(z)} \prec h(z). \tag{46}$$

By Lemma 1, (46) implies $w(z) \prec h(z).$ The remaining part of the proof is similar to that of Theorem 2, so we omit it.

Remark **.** *For* $\alpha_1 = a, \alpha_2 = 1$ *and* $\beta_1 = c, a, c > 0$ *, Theorem 6 corrected Theorem 2.5 for Oana [9].*

The proof of the following theorem is similar to that of Theorems 3 and 5, so we omit it.

Theorem 7. *Let* $h \in \mathrm{P}$ *with* $R\{\mu - ph(z)\} > 0$ *and* $f \in K_k^{\lambda}(\alpha_1, \beta_1, h),$ *with respect to* $g \in S_k^{\lambda}(\alpha_1, h),$ *then,* $F_{p,\mu}(f) \in K_k^{\lambda}(\alpha_1, \beta_1, h)$ *with respect to* $G = F_{p,\mu}(g)$ *provided* $G_k^{\lambda}(\alpha_1)(z) \neq 0.$

Note that for $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1,$ we have $Rh(z) = \frac{1+A}{1+B}.$

Remark **.** *Taking* $h(z) = \frac{1+Az}{1+Bz},$ *in Theorems 2-7, we get corresponding results for the classes* $S_k^{\lambda}(\alpha_1, \beta_1, A, B)$ *and* $K_k^{\lambda}(\alpha_1, \beta_1, A, B).$

Theorem 8. *If* $h \in \mathrm{P}$, *with* $R\{p+1-\alpha-ph(z)\}>0$, $f \in S_k^\lambda(\alpha_1,\beta_1,h), \varphi \in \sum_{p,m}$ *and* $z^{p+1}\varphi(z) \in R(\alpha), \alpha<1$, *then* $f*\varphi \in S_k^\lambda(\alpha_1,\beta_1,h)$.

Proof. For $f \in S_k^\lambda(\alpha_1,\beta_1,h)$, we have

$$F(z) = -\frac{z\left(M_{p,l,s}^{\lambda}(\alpha_1) f(z)\right)'}{pf_k^\lambda(\alpha_1)(z)} \prec h(z). \tag{47}$$

Let

$$\psi(z) = z^{p+1} f_k^\lambda(\alpha_1)(z),$$

then $\varphi \in \mathrm{A}$ and

$$\frac{z\psi'(z)}{\psi(z)} = p+1+\frac{z\left(f_k^\lambda(\alpha_1)(z)\right)'}{f_k^\lambda(\alpha_1)(z)} \prec p+1-ph(z). \tag{48}$$

From the hypotheses of the theorem, we see that

$$R\frac{z\psi'(z)}{\psi(z)} > \alpha, \tag{49}$$

that is $\psi \in S^*(\ \alpha), \alpha<1.$

For $\varphi \in \sum_{p,m}$ it is easy to get

$$z^{p+1}M_{p,l,s}^{\lambda}(\alpha_1)(f*\varphi)(\in_k^j\ z) = (z^{p+1}\varphi(z)) * M_{p,l,s}^{\lambda}(\alpha_1) f(\in_k^j\ z)$$

and

$$z^{p+2}(M_{p,l,s}^{\lambda}(\alpha_1)(f*\varphi)(z))' = (z^{p+1}\varphi(z)) * (z^{p+2}M_{p,l,s}^{\lambda}(\alpha_1) f(z))'.$$

So, we have

$$\begin{aligned}\Psi(z) &= -\frac{(M_{p,l,s}^{\lambda}(\alpha_1)(f*\varphi)(z))'}{\frac{p}{k}\sum_{j=0}^{k-1} \in_k^{jp}\ M_{p,l,s}^{\lambda}(\alpha_1)(f*\varphi)(\in_k^j\ z)} \\ &= -\frac{(z^{p+1}\varphi(z)) * z^{p+2}(M_{p,l,s}^{\lambda}(\alpha_1) f(z))'}{pz^{p+1}\varphi(z) * (z^{p+1} f_k^\lambda(\alpha_1)(z))} \\ &= \frac{z^{p+1}\varphi(z) * (\psi(z)F(z))}{z^{p+1}\varphi(z) * \psi(z)}.\end{aligned} \tag{50}$$

Since h is convex, univalent, applying Lemma 3, it follows $\Psi(z) \prec h(z)$, that is $f*\varphi \in S_k^\lambda(\alpha_1,\beta_1,h)$.

Remark 3. Specializing the parameters l, s, α_i, β_j in the above results, we obtain results corresponding to the special operators in [8].

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this paper.

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МЕРОМОРФНЫЕ ПОДКЛАССЫ p -ВАЛЕНТНЫХ ФУНКЦИЙ С УЧЕТОМ ОПЕРАТОРА

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РЕЗЮМЕ

В этой статье мы исследуем некоторые отношения включения двух новых подклассов мероморфно p-валентных функций, определенных с помощью линейного оператора. Мы также изучаем некоторые сохраняющие интегральные свойства и свойства свертки этих классов.

Ключевые слова: Аналитический, p-валентный, мероморфен, линейный оператор, дифференциальная подчиненность, включение отношения.