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# LINEAR DEGENERATE CONVOLUTION-ELLIPTIC EQUATIONS WITH PARAMETERS*

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**Abstract.** The separability properties of degenerate abstract convolution-elliptic equations are investigated. Here the considered equation contains a certain parameter and we find sufficient conditions that guarantee the separability of linear problems in weighted L_p spaces. In the present paper separability properties of degenerate convolution differential-operator equations with parameter in vector-valued weighted spaces are obtained. By using these results the existence and uniqueness of maximal regular solution of the degenerate convolution equation is obtained in weighted L_p spaces. In application, the maximal regularity properties of the Cauchy problem for degenerate abstract parabolic equation in mixed L_p norms are established.

**Keywords:** sectorial operators, abstract weighted spaces, operator-valued multipliers, degenerate convolution equations,convolution equations with parameter.

**AMS Subject Classification:** 34G10, 45J05

## 1. Introduction

In a series of recent publications regularity properties of differential operator equations, especially elliptic and parabolic type have been studied extensively e.g., in [1-4], [6,7], [10,11], [15-20], [23] and the references therein. It is well known that the differential equations with parameters play important role in modelling of physical processes. Therefore some authors are investigated this type of equations. Differential-operator equations with parameters have also significant applications in nonlinear analysis. Convolution operators in Banach-valued function spaces studied e.g., in [12-14], [16], [18]. In [12] and [20] regularity properties of degenerate convolution-differential operator equations (CDOEs) are studied.
In recent years, operator-valued Fourier multiplier theorems on diverse vector-valued function spaces have been studied (see [8], [20], [23]). Fourier multiplier theorems with operator-valued multiplier functions have found many applications also in the theory of convolution equations, in particular elliptic CDOEs. They are needed to establish existence and uniqueness as well as regularity for convolution differential-operator equations (CDOEs) in Banach spaces.
Note that, the CDOEs, in particular case, degenerate CDOEs with parameters are relatively less investigated subject. The main aim of the present paper is to study

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*Работа была представлена на семинаре Института Прикладной Математики 18.12.2018 the maximal $L_p$-regularity properties of the degenerate linear CDOEs with parameters

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^{[\alpha]}u + (A+\lambda) * u = f(x), \qquad x \in R^n, \tag{1}$$

It is clear to see that the solution u of the problem (1.1) depends on the parameter $\varepsilon$ i.e.,$u = u(\varepsilon, x)$.

In application we obtain the well-posedness of the Cauchy problem for degenerate parabolic CDOE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^{[\alpha]}u + A * u = f(t,x),$$

$$u(0,x) = 0, t \in \mathbb{R}_+, x \in R^n, \tag{2}$$

in $E$ −valued mixed $L_p$ −-spaces, where $l$ is a natural number,$a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \alpha_k$are nonnegative integers, $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n),$, $\varepsilon_\alpha = \prod_{k=1}^n \varepsilon_k^{\frac{\alpha_k}{l}}$, $\varepsilon_k$are positive, $\lambda$ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space $E$ for $x \in R^n$. Here, the convolutions $a_\alpha * D^{[\alpha]}u$, $A * u$ are defined in the distribution sense (see e.g., [2]), where $\gamma = \gamma(x)$ is a positive measurable function on $\Omega \subset R^n$ and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} \dots D_{x_n}^{[\alpha_n]}, \qquad D_{x_i}^{[\alpha_i]} = \left(\gamma(x)\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

$L_{\mathbf{p}}(R_+^{n+1}; E)$denotes the space of all **p**-summable complex-valued functions with mixed norm (see e.g., [5]), i.e., the space of all measurable functions $f$ defined on $R_+^{n+1}$, for which

$$\|f\|_{L_{\mathbf{p},\gamma}(R_+^{n+1};E)} = \left( \int_{R^n} \left( \int_{\mathbb{R}_+} \|f(t,x)\|_E^p \gamma(x) dx \right)^{\frac{p_1}{p}} dt \right)^{\frac{1}{p_1}} < \infty, R_+^{n+1}$$
$$= R^n \times \mathbb{R}_+, \boldsymbol{p} = (p, p_1).$$

One of main features of the present work is that the convolution equations are degenerate on some points of $\mathbb{R} = (-\infty, \infty)$ and the equation (1) has a certain parameter. In this paper, we establish the uniform separability properties of the parameter dependent problem (1) and the uniform maximal regularity of Cauchy problem for parabolic degenerate CDOE with parameters (2). The main tools of this work is the theory of operator-valued Fourier multipliers. Since the equation (1) has a certain parameter, some difficulties occur.

The present paper is organized as follows. The first section in this paper contains an introduction. In section 2, we introduce the required notations, definitions and basic properties of vector-valued function spaces. Section 3, contains coercive estimates on $E-$valued weighted spaces. Namely, we prove that for all $L_{p,\gamma}(R^n;E)$ there is a unique solution $u(\varepsilon,x)$ of problem (1) and corresponding coercive uniform estimate hold with respect to the spectral parameter $\lambda$. Moreover, section 3 prepares for the proof of the main result of this paper. In section 4, using the results obtained in section 3, we establish the maximal regularity of (1) and we conclude that the above Cauchy problem for the parabolic degenerate CDOE with parameters has a unique solution satisfying coercivity estimates.

**2. Notations and basic definitions**

We start by giving the notation and definitions to be used in this paper. Let us first some notions.

Let $E$ be a Banach space and $\Omega$ be a domain in $R^n$. $C(\Omega,E)$and$C^{(m)}(\Omega;E)$ will denote the spaces of $E-$valued bounded uniformly strongly continuous and $m-$times continuously differentiable functions on $\Omega$, respectively.

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi=\{\lambda;\ \lambda\in\mathbb{C},\qquad |arg\lambda|\le\varphi\}\cup\{0\},\qquad 0\le\varphi<\pi.$$

$E_1$and$E_2$be two Banach spaces and let $B(E_1,E_2)$denote the space of bounded linear operators from $E_1$ to $E_2$. For $E_1=E_2=E$we denote $B(E,E)$ by $B(E)$.

$D(A), R(A)$and $KerA$ denote the domain, range and null space of the linear operator in $E$, respectively.

$S=S(R^n;E)$denotes the E-valued Schwartz class, i.e. the space of $E-$valued rapidly decreasing smooth functions on $R^n$, equipped with its usual topology generated by seminorms. $S(R^n;\mathbb{C})$will be denoted by just $S$.

Let $S'(R^n;E)$ denote the space of all continuous linear operators, $L: S\to E$, equipped with topology of bounded convergence. Recall $S(R^n;E)$ is norm dense in $L_{p,\gamma}(R^n;E)$when $1<p<\infty,\gamma\in A_p$ (see e.g., [9]).

The weight $\gamma=\gamma(x)$ satisfy an $A_p$ condition, i.e., $\gamma\in A_p, p\in(1,\infty)$ if there is a positive constant $C$ such that

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\gamma(x)dx\right)\left(\frac{1}{|Q|}\int_Q\gamma^{-\frac{1}{p-1}}(x)dx\right)^{p-1}\le C$$

for all cubes $Q\subset R^n$ (see e.g., [9, Ch.9]).

The result [22] implies that the space $l_q$ for $q \in (1,\infty)$ satisfies multiplier condition with respect to $p \in (1,\infty)$ and the weight functions $\gamma(x) = \prod_{k=1}^{n} |x_k|^{\nu}$ for $-\frac{1}{n} < \nu < \frac{1}{n}(p-1)$.

An $E$-valued generalized function $D^{\alpha} f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S(R^n; E)$ if

$$\langle D^{\alpha} f, \varphi\rangle = (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle$$

holds for all $\varphi \in S$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i$ are integers.

Let $F$ denote the Fourier transform. Throughout this section the Fourier transformation of a function $f$ will be denoted by $\hat{f}$ and $F^{-1} f = \check{f}$. It is known that

$$F(D_x^{\alpha} f) = (i\xi_1)^{\alpha_1} \dots (i\xi_n)^{\alpha_n} \hat{f}\,, D_{\xi}^{\alpha}\big(F(f)\big) = F[(-ix_1)^{\alpha_1}, \dots (-ix_n)^{\alpha_n} f]$$

for all $f \in S'(R^n; E)$.

$L_{p,\gamma}(\Omega; E)$ denote the space of strongly $E$ −valued functions that are defined on $\Omega$ with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p \,\gamma(x) dx\right)^{\frac{1}{p}}, \ \ 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x\in\Omega} [\gamma(x)\|f(x)\|_E],$$

where, $\gamma = \gamma(x), x = (x_1, x_2, \dots, x_n)$ be a positive measurable weighted function on a measurable subset $\Omega \subset R^n$. For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$

A closed linear operator $A$ is said to be $\varphi$ −sectorial (or sectorial for $\varphi = 0$) in a Banach space $E$ with bound $M > 0$ if $Ker\, A = \{0\}, D(A)$ and $R(A)$ are dense on $E$, and $\|(A + \lambda I)^{-1}\|_{B(E)} \le M|\lambda|^{-1}$ for all $\lambda \in S_{\varphi}, \varphi \in [0, \pi)$, where $I$ is an identity operator in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by $A_{\lambda}$. It is known (see e.g., [21, §1.15.1]) that the fractional powers of the operator $A$ are well defined.

Let $E(A^{\theta})$ denote the space $D(A^{\theta})$ with the graph norm

$$\|u\|_{E(A^{\theta})} = \left(\|u\|_E^p + \left\|A^{\theta} u\right\|_E^p\right)^{\frac{1}{p}}, 1 \le p < \infty, \qquad -\infty < \theta < \infty.$$

Note that the above norms are equivalent for $p \in [1, \infty)$.

Let $A = A(x), x \in R^n$ be closed linear operator in $E$ with domain $D(A)$ independent of $x$. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined as

$$\hat{A}(\xi)u(\varphi) = A(x)u(\hat{\varphi}) \ for\ u \in S'(R^n; D(A)), \varphi \in S(R^n).$$

A function $\Psi \in L_\infty\left(R^n; B(E_1, E_2)\right)$is called a Fourier multiplier from in $L_{p,\gamma}(R^n; E_1)$to $L_{p,\gamma}(R^n; E_2)$ for $p \in (1,\infty)$ if the map $u \to Tu = F^{-1}\Psi(\xi)Fu, u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$T: L_{p,\gamma}(R^n; E_1) \to L_{p,\gamma}(R^n; E_2).$$

A set $K \subset B(E_1, E_2)$ is called $R$ −bounded (see e.g., [6], [23]) if there is a constant $C > 0$ such that for all $T_1, T_2, \dots, T_m \in K$and $u_1, u_2, \dots, u_m \in E_1, m \in \mathbb{N}$,

$$\int_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_{E_2} dy \le C \int_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy,$$

where$\{r_j\}$ is a sequence of independent symmetric $\{-1; 1\}$ −valued random variables on [0,1]. The smallest$C$ for which the above estimate holds is called the $R$ −bound of $K$ and denoted by $R(K)$.

A Banach space $E$ is said to be a space satisfying the multiplier condition with respect to weighted function $\gamma$ and $p \in (1,\infty)$ (or multiplier condition with respect to $p \in (1,\infty)$ when $\gamma(x) \equiv 1$) if for any $\Psi \in C^{(n)}(R^n\backslash\{0\}; B(E))$ the $R$ −boundedness of the set

$$\left\{|\xi|^{|\beta|} D_\xi^\beta \Psi(\xi): \xi \in R^n\backslash\{0\}, \qquad \beta = (\beta_1, \beta_2, \dots, \beta_n), \beta_k \in \{0,1\}\right\}$$

implies that $\Psi$ is a Fourier multiplier in $L_{p,\gamma}(R^n; E)$.

A sectorial operator $A(x), x \in R^n$ is said to be uniformly $R$ −sectorial in a Banach space $E$ if there exists a $\varphi \in [0,\pi)$ such that

$$\sup_{x\in R^n} R\left(\left\{[A(x)(A(x) + \xi I)^{-1}]: \xi \in S_\varphi\right\}\right) \le M.$$

Note that, in Hilbert spaces every norm bounded set is $R$ −bounded. Therefore, in Hilbert spaces all sectorial operators are $R$ −sectorial.

Let $h \in \mathbb{R}, m \in \mathbb{N}$and $e_k$, k=1,2,…,*n*be the standard unit vectors of $R^n$,

$$\Delta_k(h) f(x) = f(x + h e_k) - f(x).$$

$A(x)$is differentiable if there is the limit

$$\left(\frac{\partial A}{\partial x_k}\right) u = \lim_{h\to 0} \frac{\Delta_k(h) A(x) u}{h}, \quad k = 1,2,\dots n, u \in D(A),$$

in the sense of $E$-norm.

Let$E_0$ and $E$ be two Banach spaces, where $E_0$ is continuously and densely embedded into $E$. Let $l$ be a natural number. $W_{p,\gamma}^l(R^n; E_0, E)$denotes the space of all functions from $S'(R^n; E_0)$ such that $u \in L_{p,\gamma}(R^n; E_0)$ and the generalized derivatives$D_k^l u = \frac{\partial^l u}{\partial x_k^l} \in L_{p,\gamma}(R^n; E)$with the norm

$$\|u\|_{W^l_{p,\gamma}(R^n;E_0,E)} = \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{k=1}^{n} \left\|D^l_k u\right\|_{L_{p,\gamma}(R^n;E)} < \infty.$$

It is clear that

$$W^l_{p,\gamma}(R^n;E_0,E) = W^l_{p,\gamma}(R^n;E) \cap L_{p,\gamma}(R^n;E_0).$$

$W^{[l]}_{p,\gamma}(R^n;E_0,E)$denotes the space of all functions from$S'(R^n;E)$ such that $u \in L_p(R^n;E_0)$ and $D^{[l]}_k u \in L_p(R^n;E)$with the norm

$$\|u\|_{W^{[l]}_{p,\gamma}(R^n;E_0,E)} = \|u\|_{L_p(R^n;E_0)} + \sum_{k=1}^{n} \left\|D^{[l]}_k u\right\|_{L_p(R^n;E)} < \infty.$$

Note that if $l \geq 2, E$ is a space satisfying the multiplier condition with respect to weighted function $\gamma$ and $p \in (1,\infty)$, then the above definitions are equivalent with usual definitions (see e.g., [17]), i.e.,

$$\|u\|_{W^l_{p,\gamma}(R^n;E_0,E)} \simeq \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{|\alpha|\leq l} \|D^\alpha u\|_{L_{p,\gamma}(R^n;E)}$$

$$\|u\|_{W^{[l]}_{p,\gamma}(R^n;E_0,E)} \simeq \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{|\alpha|\leq l} \|D^\alpha u\|_{L_p(R^n;E)}$$

## 3.Linearnondegenerate convolution-elliptic equation with parameters

It is well known that Fourier multiplier theorems with operator valued multiplier functions have found many applications in the theory of CDOEs. To obtain the main result, we first consider the following equation, then apply the result obtained to solve equation (1.1).

Consider the following nondegenerate CDOE with parameters,

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + (A+\lambda) * u = f, \qquad (3)$$

whereε, $\varepsilon_\alpha$, λ are parameters, $a_\alpha$ are complex-valued functions defined in (1) and $A$ is a linear operator in a Banach space $E$.

We find sufficient conditions that guarantee the separability of the problem (3).This facts is derived by using is a Fourier multiplier theorem for operator-valued multiplier functions on vector-valued Banach spaces. We establish existence and uniqueness as well as regularity for differential equations in Banach spaces and thus also for CDOEs.

Note that, Section 3, prepares for the proof of the main result of this paper. In this section we rely on the recent paper by [16] and [20], where the problem is studied extensively. First we establish coercive estimate for nondegenerate case equation (3) and apply them to study regularity of the degenerate equation (1).

**Condition 3.1.** Suppose the following are satisfied:
1)

$$L_\varepsilon(\xi) = \sum_{|\alpha|\le l} \varepsilon_\alpha \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}, \varphi_1 \in [0,\pi) \; for \;\; \xi \in R^n,$$

$$|L_\varepsilon(\xi)| \ge C \sum_{k=1}^{n} \varepsilon_k \left|\hat{a}_{\alpha(l,k)}\right| |\varepsilon_k|^l,$$

$$\alpha(l,k) = (0,\ldots,l,0,\ldots,0), i.e. \alpha_i = 0, i \ne k, \alpha_k = l;$$

2) $\hat{a}_\alpha \in C^{(n)}(R^n)$,and$|\xi|^{|\beta|}\left|D^\beta \hat{a}_\alpha(\xi)\right| \le C_1, \beta_k \in \{0,1\}, 0 \le |\beta| \le n$;
3) $\left[D^\beta \hat{A}(\xi)\right]\hat{A}^{-1}(\xi_0) \in C\left(R^n; B(E)\right), |\xi|^{|\beta|}\left\|\left[D^\beta \hat{A}(\xi)\right]\hat{A}^{-1}(\xi_0)\right\|_{B(E)} \le C_2$ for$0 \le |\beta| \le n, \xi, \xi_0 \in R^n \backslash \{0\}$.
Let $X = L_{p,\gamma}(R^n;E), Y = W^l_{p,\gamma}(R^n;E(A),E), p \in (1,\infty)$.

**Theorem 3.1.** Assume that Condition 3.1 holds and $E$ is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1,\infty)$. Let $A$ be a uniformly $R$ −sectorial operator in $E$ with $\varphi \in [0,\pi), \lambda \in S_{\varphi_2}$ and $0 \le \varphi + \varphi_1 + \varphi_2 < \pi$.Then, problem (3) has a unique solution $u$ and the coercive uniform estimate holds

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X + \|A * u\|_X + |\lambda|\|u\|_X \le C\|f\|_X, \tag{4}$$

for all $f \in X$ and $\lambda \in S_\varphi$ .

**Proof.** By applying the Fourier transform to equation (3) we get

$$\hat{u}(\xi) = \left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1} \hat{f}(\xi). \tag{5}$$

Hence, the solution of (3) can be represented as $u(x) = F^{-1}\left[\hat{A}(\xi) + \lambda + L_\varepsilon(\xi)\right]^{-1}\hat{f}$ and there are positive constants $C_1$ and $C_2$ such that

$$C_1|\lambda|\|u\|_X \le \left\|F^{-1}\left[\lambda\left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1}\right]\hat{f}\right\|_X \le C_2\lambda|\|u\|_X,$$

$$C_1\|A * u\|_X \le \left\|F^{-1}\left[\hat{A}(\xi)\left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1}\right]\hat{f}\right\|_X \le C_2\|A * u\|_X,$$

$$C_1 \sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X$$
$$\leq \left\| F^{-1}\left[\sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha \left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1}\right] \hat{f} \right\|_X$$
$$\leq C_2 \sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X. \quad (6)$$

By using a similar technique as in [13] and [20], definition of $R-$boundedness we obtain the operator-valued functions, arising in the solution of equation (3) are Fourier multipliers from $X$to $X$. Thus, from (5) and (6) we obtain

$$|\lambda| \|u\|_X \leq C_0 \|f\|_X, \|A * u\|_X \leq C_1 \ \ \|f\|_X, \sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X \leq C_2 \|f\|_X,$$

for all $f \in X$. Hence, we get the assertion.

Let $O_\varepsilon$ be an operator in $X$ generated by problem (3.1) for $\lambda = 0$, i.e.,

$$D(O_\varepsilon) \subset Y, O_\varepsilon u = \sum_{|\alpha|\leq l} \varepsilon_\alpha \, a_\alpha * D^\alpha u + A * u.$$

From Theorem 3.1 we have:

**Result 3.1.** Assume that the all conditions of Theorem 3.1 hold. Then, for all $\lambda \in S_{\varphi_2}$the following uniform coercive estimate holds

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha \, |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha (O_\varepsilon + \lambda)^{-1}\|_{B(X)} +$$
$$+\|A * (O_\varepsilon + \lambda)^{-1}\|_{B(X)} + \|\lambda (O_\varepsilon + \lambda)^{-1}\|_{B(X)} \leq C.$$

## 4. Degenerate convolution-elliptic equations

As we mentoined before, from the view point of differential operator equations with parameters, vector-valued Banach spaces form one class of function spaces which are of special interest.In this section we rely on the recent papers by [12] and [16].

Let us consider the following degenerate elliptic CDOE with parameters

$$\sum_{|\alpha|\le l} \varepsilon_\alpha a_\alpha * D^{[\alpha]}u + A * u + \lambda u = f, \qquad (7)$$

in weighted space $L_{p,\gamma}(R^n;E)$, where, $l$ is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha_k$ are nonnegative integers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $\varepsilon_\alpha = \prod_{k=1}^{n} \varepsilon_k^{\frac{\alpha_k}{l}}$, $\varepsilon_k$are positive, $\lambda$ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space $E$ for $x \in R^n$. Here, the convolutions $a_\alpha * D^{[\alpha]}u,\ A * u$ are defined in the distribution sense (see e.g., [2]), where $\gamma = \gamma(x)$ is a positive measurable function on $\Omega \subset R^n$and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} \dots D_{x_n}^{[\alpha_n]}, \qquad D_{x_i}^{[\alpha_i]} = \left(\gamma(x)\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

For this purpose consider the following substitution

$$y_k = \int_0^{x_k} \gamma^{-1}(z)dz, \quad k = 1,2,\dots,n. \qquad (8)$$

It is clear that, under the substitution (8), $D^{[\alpha]}u$ transforms to $D^\alpha u$. Moreover, the spaces$L_p(R^n;E), W_{p,\gamma}^{[l]}(R^n;E(A),E)$are mapped isomorphically onto the weighted spaces $L_{p,\gamma}(R^n;E)$ and $W_{p,\gamma}^{l}(R^n;E(A),E)$ respectively where,

$$\gamma = \tilde{\gamma}(y) = \gamma(x(y)) = \gamma(x_1(y_1), x_2(y_2), \dots, x_n(y_n)).$$

Moreover, under (8) the degenerate problem (7) considered in $L_p(R^n;E)$is transformed into the nondegenerate problem (3) in $L_{p,\gamma}(R^n;E)$, where

$$a_\alpha = a_\alpha(y) = a_\alpha(\tilde{\gamma}(y)), \qquad u = u(y) = \tilde{u}(y) = u(\tilde{\gamma}(y))$$
$$A = A(y) = \tilde{A}(y) = A(\tilde{\gamma}(y)), f = f(y) = \tilde{f}(y) = f(\tilde{\gamma}(y)).$$

Let $\tilde{X} = L_p(R^n;E), \tilde{Y} = W_{p,\gamma}^{[l]}(R^n;E(A),E),\ p \in (1,\infty)$.

In this section we show the following result:

**Theorem 4.1.** Assume that Condition 3.1 and substitution (8) holds for $a_\alpha = a_\alpha(y)$ and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1,\infty)$. Let $A$ be a uniformly $R$ −sectorial operator in $E$ with $\varphi \in [0,\pi), \lambda \in S_{\varphi_2}$ and $0 \le \varphi + \varphi_1 + \varphi_2 < \pi$ for $A = A(y)$. Then for all $f \in \tilde{X}$ there is a unique solution of the problem (7) and the following coercive uniform estimate holds:

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]} \right\|_{\tilde{X}} + \|A * u\|_{\tilde{X}} + |\lambda| \|u\|_{\tilde{X}}$$
$$\leq C\|f\|_{\tilde{X}}. \tag{9}$$

**Proof.** By substitution (8), the degenerate problem (7) is transformed into the nondegenerate problem (3) considered in the weighted space $L_{p,\gamma}(R^n; E)$ Then in view of Theorem 3.1 we obtain the assertion.

Let$Q_\varepsilon$ be the operator generated by problem (7) i.e.,

$$D(Q_\varepsilon) = W_{p,\gamma}^{[l]}(R^n; E(A), E), \qquad Q_\varepsilon u = \sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^{[\alpha]} u + A * u.$$

From the Theorem 4.1 and Result 3.1.we obtain the for $\lambda \in S_\varphi$ there exist the resolvent of operator $Q_\varepsilon$ and has the estimate

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]} (Q_\varepsilon + \lambda)^{-1} \right\|_{B(\tilde{X})} +$$
$$+\|A * (Q_\varepsilon + \lambda)^{-1}\|_{B(\tilde{X})} + \|\lambda (Q_\varepsilon + \lambda)^{-1}\|_{B(\tilde{X})} \leq C$$

It is known that Fourier multipliers theorems with operator-valued multiplier functions have found many applications in the theory of convolution equations. For this purpose, as an application we consider the following Cauchy problem for the degenerate parabolic CDOE,

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^{[\alpha]} u + A * u + du = f(t, x),$$
$$u(0, x) = 0, \ \ t \in \mathbb{R}_+, x \in R^n, \tag{10}$$

in$E$ $-$valued mixed $L_{\mathbf{p},\gamma}(R_+^{n+1}; E)$-space, where $\varepsilon, \varepsilon_\alpha$ are parameters, $d > 0$, $a_\alpha$are complex-valued functions defined in (1) and $A$ is a linear operator in a Banach space $E$.

$L_{\mathbf{p},\gamma}(R_+^{n+1}; E)$denote the space of all $\boldsymbol{p}$ $-$summable complex-valued functions with mixed norm (see e.g., [5]), i.e., the space of all measurable functions $f$ defined, on $R_+^{n+1}$, for which

$$\|f\|_{L_{p,\gamma}(R_+^{n+1}; E)} = \left( \int_{R^n} \left( \int_{\mathbb{R}_+} \|f(t, x)\|_E^p \gamma(x) dx \right)^{\frac{p_1}{p}} dt \right)^{\frac{1}{p_1}} < \infty, R_+^{n+1}$$
$$= R^n \times \mathbb{R}_+, \boldsymbol{p} = (p, p_1).$$

By using a similar technique as in [13] and [17] we obtain the problem (10) has a unique solution $u(\varepsilon, t, x)$ and for sufficiently large $d$ the following coercive uniform estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_p(R_+^{n+1};E)} + \sum_{|\alpha|\leq l} \varepsilon_\alpha \left\|a_\alpha * D^{[\alpha]}u\right\|_{L_p(R_+^{n+1};E)} + \|A*u\|_{L_p(R_+^{n+1};E)} \leq C\|f\|_{L_p(R_+^{n+1};E)}.$$

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**Линейно вырождающиеся сверточно-эллиптические уравнения с параметрами**

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## РЕЗЮМЕ

Исследуются свойства сепарабельности вырожденных абстрактных сверточно-эллиптических уравнений. Здесь рассматриваемое уравнение содержит некоторый параметр и мы находим достаточные условия, гарантирующие сепарабельность линейных задач в весовых Lp -пространствах. В настоящей статье получены свойства сепарабельности вырождающихся сверточных дифференциально-операторных уравнений с параметром в векторнозначных весовых пространствах. Используя эти результаты, в весовых Lp -пространствах получается существование и единственность максимально-регулярного решения вырожденного сверточного уравнения. В приложении установлены максимальные регулярности задачи Коши для вырожденного абстрактного параболического уравнения в смешанных Lp-нормах.

**Ключевые слова:**секториальные операторы, абстрактные весовые пространства, операторнозначные мультипликаторы, вырожденные уравнения свертки, уравнения свертки с параметром.