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BEST APPROXIMATION OF THE CONJUGATE FUNCTIONS IN MORREY SPACES

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Abstract. We investigate the best approximation of the conjugate functions in Morrey spaces. The best approximation of the conjugate functions have been obtained in terms of the best approximation of the functions. Some direct and inverse theorems for approximation by trigonometric polynomials in Morrey spaces are proved. The modulus of smoothness of conjugate functions were estimated in terms of the best approximation of the given functions.

Keywords: Morrey spaces, modulus of smoothness, direct theorems, best approximation, inverse theorem.

AMS Subject Classification: 41A10, 41A50, 42A10.

1. Introduction

Let T denotes the interval $[0,2\pi]$. Let $L^p(T)$, $1\le p<\infty$ be the Lebesgue space of all measurable 2π - periodic functions defined on T such that

$$\|f\|_p := \left(\int_T |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty .$$

The *Morrey spaces* $L_0^{p,\lambda}(T)$ for a given $0\le\lambda\le 1$ and $p\ge 1$, we define as the set of functions $f\in L_{loc}^p(T)$ such that

$$\|f\|_{L_0^{p,\lambda}(T)} := \left\{\sup_I \frac{1}{|I|^{1-\lambda}} \int_I |f(t)|^p\right\}^{\frac{1}{p}} < \infty$$

where the supremum is taken over all intervals $I\subset[0,2\pi]$. Note that $L_0^{p,\lambda}(T)$ becomes a Banach spaces, $\lambda=1$ coincides with $L^p(T)$ and for $\lambda=0$ with

$L^{\infty}(T)$. If , $0 \le \lambda_1 \le \lambda_2 \le 1$, then $L_0^{p,\lambda_2}(T) \subset L_0^{p,\lambda_1}(T)$. Also, if $f \in L_0^p(T)$, then $f \in L^p(T)$ and hence $f \in L^p(T)$. The Morrey spaces, were introduced by C. B. Morrey in 1938. The properties of the these spaces have been investigated intensively by several authors and together with weighted Lebesgue spaces $L_{\omega}^p(T)$ play an important role in the theory of partial equations, in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in [14-17], [30], [33], [35], [37], [38] and [44].

In this study we will use the following notations:
$Z^+ := \{1,2,3,...\}, N_0 := Z^+ \cup \{0\}$. Also, we shall use $c, c_1, c_2, ...$ to denote constants depending only on parameters that are not important for the questions of our interest

Denote by $C^{\infty}(T)$ the set of all functions that are realized as the restriction to T of elements in $C^{\infty}(R)$. Also we define $L^{p,\lambda}(T)$ to be closure of $C^{\infty}(T)$ in $L_0^{p,\lambda}(T)$.

Note that in this study we investigate the direct and inverse problems of approximation theory in Morrey space $L^{p,\lambda}(T)$, the closure of the set of trigonometric polynomials in $L_0^{p,\lambda}(T)$ with $1 < p < \infty$.
The function

$$\omega_{p,\lambda}^{\alpha}(f,\delta) := \sup_{|h| \le \delta} \left\| \Delta_h^{\alpha}(f,\cdot) \right\|_{L^{p,\lambda}(T)}, \ \alpha \in Z^+$$

is called α -th modulus of smoothness $f \in L^{p,\lambda}(T)$, $0 \le \lambda \le 1$ and $p \ge 1$, where

$$\Delta_h^{\alpha}(f,\cdot) = \sum_{k=0}^{\infty} (-1)^{\alpha-k} \binom{\alpha}{k} f(x+kh), \ \alpha \in Z^+.$$

The modulus of smoothness $\omega_{L^{p,\lambda}(T)}^{\alpha}(f,\delta)$ have the following properties [23]:

1) $\omega_{p,\lambda}^{\alpha}(f,\delta)$ is an increasing function,
2) $\lim_{\delta \to 0} \omega_{p,\lambda}^{\alpha}(f,\delta) = 0$ for every $f \in L^{p,\lambda}(T)$, $0 \le \lambda \le 1$ and $p \ge 1$,

3) $\omega_{p,\lambda}^{\alpha}\left(f+g,\delta\right)\leq\omega_{p,\lambda}^{\alpha}\left(f,\delta\right)+\omega_{p,\lambda}^{\alpha}\left(f,\delta\right)$ for $f,g\in L^{p,\lambda}\left(T\right)$,

4) $\omega_{p,\lambda}^{\alpha}(f,n\delta)\leq n^{\alpha}\omega_{p,\lambda}^{\alpha}(f,\delta),\ n\in N_0$

5) $\omega_{p,\lambda}^{\alpha}\left(f,s\delta\right)\leq\left(s+1\right)^{\alpha}\omega_{p,\lambda}^{\alpha}\left(f,\delta\right),\ s>0$,

6) $\omega_{p,\lambda}^{\alpha}(f,\delta)\leq[(n+1)\delta+1]^{\alpha}\omega_{p,\lambda}^{\alpha}(f,\frac{1}{n+1}),\ n\in N_0$

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k\left(x,f\right) \tag{1}$$

be the Fourier series of the function $f\in L_1\left(T\right)$, where $A_k\left(x,f\right)=a_k\left(f\right)\cos kx+b_k\left(f\right)\sin kx,\ a_k\left(f\right)$ and $b_k\left(f\right)$ are Fourier coefficients of the function $f\in L_1\left(T\right)$.

Let $S_n\left(f\right),\ \left(n=1,2,...\right)$ be the n-*th partial sums* of the Fourier series (1) , i. e.

$$S_n\left(x,f\right)=\frac{a_0}{2}+\sum_{k=1}^{n}A_k\left(x,f\right).$$

We denote by $E_n\left(f\right)_{L^{p,\lambda}(T)}$, $\left(n=0,1,2,...\right)$ the best approximation of $f\in L^{p,\lambda}\left(T\right)$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n\left(f\right)_{L^{p,\lambda}(T)}:=\inf\left\{\left\|f-T_n\right\|_{L^{p,\lambda}(T)}:T_n\in\Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

In the proof of the main results, we use the following theorem.

Theorem 1. [22]. Let $f\in L^{p,\lambda}\left(T\right),\ 0<\lambda\leq1$ and $1<p<\infty$. Then the estimate

$$\omega_{p,\lambda}^{\alpha}\left(f,\frac{1}{n}\right)\leq\frac{c_1}{n^{\alpha}}\sum_{s=0}^{n}\left(s+1\right)^{\alpha-1}E_s\left(f\right)_{L^{p,\lambda}(T)} \tag{2}$$

holds with a constant c_1 independent of n.

2. Main Results

The problems of approximation theory in the weighted and non-weighted Morrey spaces have been investigated in [4-6], [8], [9], [18], [22], [23], [28], [32] and [36]. In this study, we investigate the problem of the best approximation for functions of a subspace of the Morrey spaces. We prove an inverse theorem of approximation theory in Morrey spaces. Similar problems in different spaces have been investigated in [1-3], [7], [10-13], [19-21], [24-27], [29], [31],[34] and [39-43].

Our main results are the following

Theorem 2. Let $f \in L^{p,\lambda}(T),\ 0<\lambda\le 1$ and $1<p<\infty$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3}$$

be the Fourier series and let

$$\sum_{n=1}^{\infty} E_n(f)_{L^{p,\lambda}(T)}\, n^{\beta-1} < \infty \tag{4}$$

where $\beta \in R$. Then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} n^{\beta} (a_n \cos nx + b_n \sin nx) \tag{5}$$

is the Fourier series of the function $\tilde{f} \in L^{p,\lambda}(T),\ 0<\lambda\le 1$ and $1<p<\infty$ and for every $\tilde{f} \in L^{p,\lambda}(T)$ the estimates

$$E_n(\tilde{f})_{L^{p,\lambda}(T)} \le c_2 \left[E_n(f)_{L^{p,\lambda}(T)}\, n^{\beta} + \sum_{k=n+1}^{\infty} E_k(f)_{L^{p,\lambda}(T)}\, k^{\beta-1} \right],\ n=1,2,..., \tag{6}$$

$$E_0(\tilde{f})_{L^{p,\lambda}(T)} \le c_3 \left[E_0(f)_{L^{p,\lambda}(T)} + \sum_{k=1}^{\infty} E_k(f)_{L^{p,\lambda}(T)}\, k^{\beta-1} \right],\ n=1,2,..., \tag{7}$$

hold with the constants $c_2,\ c_3>0$, nondependent of f and n.

Corollary 1. Under the condition of Theorem 1 the estimate

$$\omega_{p,\lambda}^{\alpha}(\tilde{f},\delta) \le c_4 \left\{ \frac{1}{n^{\alpha}} \sum_{s=0}^{n} (s+1)^{\alpha+\beta-1} E_s(f)_{L^{p,\lambda}(T)} + \sum_{k=n+1}^{\infty} k^{\beta-1} E_k(f)_{L^{p,\lambda}(T)} \right\}$$

holds with a constant $c_4>0$ depending on α and r.

Note that similar estimates in the different spaces for modulus of continuity were proved in [1], [2], [10], [24], [27], [41] and [43].

3. Proofs of the Main Results

Proof of Theorem 2. Let s_n and $\tilde{s}_n$ be the partial sums of (3) and (5) respectively and $\mu_n = n^{\beta}\ (n=1,2,...)$. By Abel transform

$$\tilde{s}_m - f = \sum_{i=1}^{m-1}(s_i - f)\Delta\mu_i + (s_m - f) \quad m = 1,2,...,$$

where $\Delta\mu_i = \mu_i - \mu_{i+1}$. It is clear that $|\Delta\mu_i| \le ci^{\alpha-1}$. For a fixed $n = 1,2,...$ and for every $k = 0,1,...$ we obtain

$$\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n} = \sum_{i=2^k n}^{2^{k+1}n-1}(s_i - f)\Delta\mu_i + \left(s_{2^{k+1}n} - f\right)\mu_{2^{k+1}n} - \left(s_{2^k n} - f\right)\mu_{2^k n}. \quad 8)$$

According to [22] the inequality

$$\|f - s_n\|_{L^{p,\lambda}(T)} \le c_5 E_n(f)_{L^{p,\lambda}(T)} \quad (9)$$

holds. Then from (8) and (9) we get

$$\left\|\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n}\right\|_{L^{p,\lambda}(T)} \le c_6 \sum_{l=2^k n}^{2^{k+1}n-1} E_l(f)_{L^{p,\lambda}(T)}\, l^{\beta-1} + c_7 E_{2^k n}(f)_{L^{p,\lambda}(T)} \left(2^k n\right)^{\beta}$$

$$\le c_8 2^k n E_{2^k n}(f)_{L^{p,\lambda}(T)} \left(2^k n\right)^{\beta-1} + c_9 \left(2^k n\right)^{\beta} E_{2^k n}(f)_{L^{p,\lambda}(T)}$$

$$= c_{10}\left(2^k n\right)^{\beta} E_{2^k n}(f)_{L^{p,\lambda}(T)}. \quad (10)$$

The inequality (10) yields

$$\sum_{k=0}^{\infty}\left\|\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n}\right\|_{L^{p,\lambda}(T)} \le c_{11}\sum_{k=0}^{\infty}\left(2^k n\right)^{\beta} E_{2^k n}(f)_{L^{p,\lambda}(T)}. \quad (11)$$

On the other hand the following inequality holds:

$$\sum_{k=1}^{\infty}\left(2^k n\right)^{\beta} E_{2^k n}(f)_{L^{p,\lambda}(T)} \le \sum_{k=n+1}^{\infty} k^{\beta-1} E_k(f)_{L^{p,\lambda}(T)}. \quad (12)$$

Using (11) and (12) we have

$$\sum_{k=0}^{\infty}\left\|\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n}\right\|_{L^{p,\lambda}(T)} \le c_{13}\left[n^{\beta} E_n(f)_{L^{p,\lambda}(T)} + \sum_{k=n+1}^{\infty} k^{\beta-1} E_k(f)_{L^{p,\lambda}(T)}\right]. \quad (13)$$

By (4), it follows that the series

$$\tilde{s}_n + \sum_{k=0}^{\infty}\left(\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n}\right)\Delta\mu_i + (s_m - f)$$

converges in the sense of the metric $L^{p,\lambda}$ to some function $\tilde{f} \in L^{p,\lambda}$. It is clear that the series (5) is the Fourier series of the function $\tilde{f}$. We can write the following inequality

$$E_n\left(\tilde{f}\right)_{L^{p,\lambda}(T)} \le \left\|\tilde{f} - \tilde{s}_n\right\|_{L^{p,\lambda}(T)} \le \sum_{k=0}^{\infty}\left\|\tilde{s}_{2^{k+1}n} - \tilde{s}_{2^k n}\right\|_{L^{p,\lambda}(T)}.$$

Now combining (13) and last relation we obtain the inequality (6) of Theorem 1.

Now we estimate $E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)}$. The inequality

$$E_n\left(\tilde{f}\right)_{L^{p,\lambda}(T)} \le \left\|\tilde{f}-\frac{a_0}{2}\right\|_{L^{p,\lambda}(T)} \le \left\|\tilde{f}-\tilde{s}_1\right\|_{L^{p,\lambda}(T)} + \left\|\tilde{s}_1-\frac{a_0}{2}\right\|_{L^{p,\lambda}(T)} \tag{14}$$

holds. From (9) and (6) we have

$$\left\|\tilde{f}-\tilde{s}_1\right\|_{L^{p,\lambda}(T)} \le c_{14} E_1\left(\tilde{f}\right)_{L^{p,\lambda}(T)} \le c_{15}\left[E_1(f)_{L^{p,\lambda}(T)} + \sum_{k=2}^{\infty} E_k(f)_{L^{p,\lambda}(T)} k^{\beta-1}\right]. \tag{15}$$

It is know that

$$\left\|\tilde{s}_1-\frac{a_0}{2}\right\|_{L^{p,\lambda}(T)} = \left\|a_1\cos x + b_1 \sin x\right\|_{L^{p,\lambda}(T)} \le 2\pi\left(\left|a_1\right|+\left|b_1\right|\right). \tag{16}$$

We chose the number t_0, such that $\left\|\tilde{f}-t_0\right\|_{L^{p,\lambda}(T)} = E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)}$. Then we obtain

$$\begin{aligned}\pi\left|a_1\right| &= \left|\int_0^{2\pi}\tilde{f}(x)\cos x dx\right| = \left|\int_0^{2\pi}\left[\tilde{f}(x)-t_0\right]\cos x dx\right| \\ &\le c_{16}\left\|\tilde{f}-t_0\right\|_{L^{p,\lambda}(T)} = c_{16}E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)}.\end{aligned} \tag{17}$$

From inequality (17) we have

$$\left|a_1\right| \le \frac{c_{16}}{\pi}E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)}. \tag{18}$$

Similar to the above, we obtain

$$\left|b_1\right| \le \frac{c_{17}}{\pi}E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)}. \tag{19}$$

Using (14) – (19), we obtain the inequality (7) of Theorem 1.

Proof of Corollary 1. Taking the relations (2), (6) and (7) into account we get

$$\omega_{p,\lambda}^{\alpha}\left(\tilde{f},\delta\right) \le \frac{c_{18}}{n^{\alpha}}\sum_{s=0}^{n}(s+1)^{\alpha-1}E_s\left(\tilde{f}\right)_{L^{p,\lambda}(T)}$$

$$= \frac{c_{19}}{n^{\alpha}}\left\{E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)} + \sum_{s=1}^{n}(s+1)^{\alpha-1}E_s\left(\tilde{f}\right)_{L^{p,\lambda}(T)}\right\}$$

$$\le \frac{c_{20}}{n^{\alpha}}\left[E_0\left(\tilde{f}\right)_{L^{p,\lambda}(T)} + \sum_{s=1}^{\infty}s^{\beta-1}E_s\left(\tilde{f}\right)_{L^{p,\lambda}(T)}\right]$$

$$+ \frac{c_{21}}{n^{\alpha}}\sum_{s=1}^{n}(s+1)^{\alpha-1}\left[E_s(f)_{L^{p,\lambda}(T)} + \sum_{k=s}^{\infty}k^{\beta-1}E_k\left(\tilde{f}\right)_{L^{p,\lambda}(T)}\right]$$

$$\leq \frac{c_{22}}{n^{\alpha}}\left[\sum_{s=0}^{n}(s+1)^{\alpha+\beta-1}E_s(f)_{L^{p,\lambda}(T)}+\sum_{s=1}^{n}(s+1)^{2\alpha-1}\sum_{k=s}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}\right]$$

$$\leq \frac{c_{23}}{n^{\alpha}}\left[\sum_{s=0}^{n}(s+1)^{\alpha+\beta-1}E_s(f)_{L^{p,\lambda}(T)}\right]$$

$$+\frac{c_{24}}{n^{\alpha}}\sum_{s=1}^{n}(s+1)^{\alpha-1}\left[\sum_{k=\nu}^{n}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}+\sum_{k=n+1}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}\right]$$

$$\leq c_{25}\left\{\frac{1}{n^{\alpha}}\sum_{s=0}^{n}(s+1)^{\alpha+\beta-1}E_s(f)_{L^{p,\lambda}(T)}+\frac{1}{n^{2\alpha}}\sum_{k=n+1}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}\sum_{l=1}^{k}l^{\beta-1}\right\}$$

$$+\sum_{k=n+1}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}$$

$$\leq c_{26}\left\{\frac{1}{n^{\alpha}}\sum_{s=0}^{n}(s+1)^{\alpha+\beta-1}E_s(f)_{L^{p,\lambda}(T)}+\frac{1}{n^{2\alpha}}\sum_{k=1}^{\infty}k^{\alpha+\beta-1}E_k(f)_{L^{p,\lambda}(T)}\right\}$$

$$+\sum_{k=n+1}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}$$

$$\leq c_{27}\left\{\frac{1}{n^{\alpha}}\sum_{s=0}^{n}(s+1)^{\alpha+\beta-1}E_s(f)_{L^{p,\lambda}(T)}+\sum_{k=n+1}^{\infty}k^{\beta-1}E_k(f)_{L^{p,\lambda}(T)}\right\}.$$

The proof of Corollary 1 is completed.

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