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CLOSENESS AND RESIDUAL CLOSENESS OF LOLLIPOP GRAPHS AND LINE GRAPHS

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Abstract. For a vertex u of the graph G, the closeness of u in G is defined as $C_G(u) = \sum_{v \in V(G)-\{u\}} 2^{-d_G(u,v)}$, which is also known as a decay centrality of u in G. The closeness of a graph $G(V,E)$ is denoted by $C(G)$ and defined by $C(G) = \sum_{u \in V(G)} C_G(u) = \sum_{u \in V(G)} \sum_{v \in V(G)-\{u\}} 2^{-d_G(u,v)}$. The residual closeness is a concept of closeness is denoted by $R(G)$ and defined by $R(G) = \min\{C(G-u) : u \in V(G)\}$, where $G-u$ is the graph obtained from G by deleting vertex u (and its incident edges). In this paper, we studied closeness and residual closeness on the lollipop graph and its line graph.

Keywords: closeness, Line graph, Lollipop graph, residual closeness, vulnerability.

AMS Subject Classification: 05C07, 05C12, 05C40, 68M10, 68R10.

1. Introduction

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of networks whose vertices represent the components of the system and the edges represent connection between a pair of vertices that enable mutual communication. The vulnerability of a communication network measures the resistance of network to the disruption of operation after the failure of certain stations or communication links. For any communication network greater degrees of stability or less vulnerability is required. Vulnerability can be measured by certain parameters like domination, connectivity, betweenness, binding number, toughness, scattering number, integrity etc. Closeness and residual closeness values are the graph parameters that measure the vulnerability and robustness of a given graph. To emphasize the importance of the closeness centrality parameter in practice, giving some examples of how this parameter is associated with real-world problems is essential. For instance, in diffusive systems, it effectively conveys the idea that high-degree nodes are important for resolving congestion at bottlenecks; in a protein structural network, it is used as a helpful measure to identify critical residues with statistically significant predictions and to discover new candidate genes for the disease [14].

Closeness can be considered as a measure of how long it will spread information from a given node to other reachable nodes in the network. By calculating the closeness and vertex residula closeness for some real networks useful results can be achieved. Residual closeness is also similar version of the closeness value that is measured the most critical node in the graph after its removal. First notable closeness definition introduced in [9]. But, it can not apply to disconnected graphs. Another closeness definition is given by Latora and Marchiori [15]. According to this definition closeness of a vertex i is $C(i) = \sum_{j \neq i} \frac{1}{d(i,j)}$. This new definition allows to be applied to disconnected graphs as well. Subsequently, Danglachev [6] revised Latora and Marchiori's definition to make the calculation easier as $C(G) = \sum_i \sum_{j \neq i} \frac{1}{2^{d(i,j)}}$. The closeness of the graph is defined as $C = \sum_i C(i)$. Residual vertex and edge closeness parameters calculate closeness value of a graph after vertex or edge removal from a graph. The consept of residual closeness based on definition of closeness and presented by Dangalchev again [6]. Closeness value of the graph is denoted by $C_k = \sum_i \sum_{j \neq i} \frac{1}{2^{d(i,j)}}$ where $d_k(i,j)$ is distance between vertices i and j after vertex k is removed from the graph. Then vertex residual closeness, denoted by R, defined as $R = min_k\{C_k\}$. In his study, Dangalchev first determined closeness centrality values for fundamental graph structures such as complete graphs, stars, paths and cycles. He also expressed the closeness value of a new graph resulting from certain graph operations in terms of closeness values of the original graphs G_1 and G_2. He also calculated the values for closeness in the Thorn graph structure [8]. In Aytaç and Gölpek [11] obtained closeness value of Tadpole graph and Mycielski graph has taken into consideration depending on diameter of original graph. Residual closeness for Helm and Sunflower graphs and cycle related graphs studied in [2] and [18], respectively. Closeness and residual closeness for Banana tress is discussed by H. Tuncel Gölpek [10]. Additionally, in Gölpek [12] focused on closeness parameter for several tree models. Relationship between closeness, residual closeness and degree, connectivity, betweenness is examined in [5] and [6]. Also, closeness for some splitting graphs and residual closeness for Mycielski graphs is calculated in [3] and [17], respectively.

In this paper, the graph G is taken as a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u,v)$ between two vertices u and v in G is the length of a shortest path joining them if any; otherwise $d(u,v) = \infty$. A shortest $u - v$ path is often called a *geodesic*. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbor vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [13]. Let u be a vertex of a graph $G = (V,E)$. Then $N(u) = \{\, v \in V(G), v\ and\ u\ are\ adjacent\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is max $d(u,v)$ for all u in G. The *radius* $r(G)$ is the minimum eccentricity of the vertices. Note that the maximum eccentricity is the diameter. A vertex v is a central vertex if $e(v) = r(G)$, and the *center* of G is the set of all central vertices [13]. The connectivity $\mathrm{K} = \mathrm{K}(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Since lollipop graphs are structures that contain both cycles and paths, the closeness and residual closeness values computed for lollipops may be useful in studying other graph families. Additionally, the number of edges in complete graphs is $\frac{n(n-1)}{2}$ times the number of vertices. So, number of edges and vertices of line graph is more than lollipop graph. As the number of edges increases, so does the connectivity and robustness. In network structures where the relationships between vertices are modeled as edges, line graphs in which the edges become vertices are very important for graph theory. In addition to the closeness value, which is one of the important parameters in measuring the stability of a graph, calculating the residual closeness value is also important in determining the critical vertex that provide information about how much a change in the closeness value of the graph by removing this vertex would cause.

This paper is organized as follows: Section 2 is devoted to some known results about closeness and residual closeness of some basic graph structures. Section 3 and 4 are about closeness and residual closeness of the lollipop graph $L_{m,n}$ and its line graph $L(L_{m,n})$, respectively. Finally, the conclusion section is presented.

2. Known Results

The closeness of a graph is defined as $C = \sum_i C(i)$, where $C(i)$ is the closeness of a vertex i, and defined as $C = \sum_{j \neq i} \frac{1}{2^{d(i,j)}}$. We can also use this definition for not connected graphs.

Let $d_k(i,j)$ be the distance between vertices i and j in the graph, received from the original graph where all link of vertex k are deleted. Then the closeness after removing vertex k is defined as $C_k = \sum_i \sum_{j \neq i} \frac{1}{2^{d_k(i,j)}}$. This definition can also be used for disconnected graphs. So, the vertex residual closeness (VRC) of a graph is defined as $R = min_k\{C_k\}$.

Theorem 2.1. [6] The closeness of

(a) the complete graph K_n with n vertices is $C(K_n) = \frac{n(n-1)}{2}$;

(b) the star graph S_n with n vertices is $C(S_n) = \frac{(n-1)(n+2)}{4}$;

(c) the path P_n with n vertices is $C(P_n) = 2n - 4 + \frac{1}{2^{n-2}}$;

(d) the cycle C_n with n vertices is

$$C(C_n) = \begin{cases} 2n(1 - 1/2^{(n-1)/2}) & ,if\ n\ is\ odd \\ n(2 - 3/2^{n/2}) & ,if\ n\ is\ even \end{cases}$$

Theorem 2.2. [6] The VRC of

(a) the complete graph K_n with n vertices is $R(K_n) = \frac{(n-1)(n-2)}{2}$;
(b) the star graph S_n with n vertices is $R(S_n) = 0$;
(c) the cycle C_n with n vertices is $R(C_n) = 2n - 6 + \frac{1}{2^{n-3}}$.

Theorem 2.3. [6] If H is a proper subgraph of graph G, then $R(H) < R(G)$.

Theorem 2.4. [7] If a vertex k does not belong to any unique geodesic (shortest path) of graph G, then $C(G\backslash k) = C(G) - 2C(k)$.

Theorem 2.5. [1] Let G be a graph. Then, for an endvertex u of G, $C_u(G) = C(G) - 2C(u)$.

Theorem 2.6. [1] If a vertex v has eccentricity two in G, then $C(v) = (\,|V(G)| + \deg(v) - 1\,)/4$.

Theorem 2.7. [1] Let G be a graph and $\{u, v\} \in V(G)$. If u is an endvertex of the support vertex v in G, then $C_v(u) = 0$.

3. Closeness of Lollipop Graph and Line Graph

In this section, we will give the definitions of Lollipop graph $L_{m,n}$ and line graph. Then, we will calculate closeness value of $L_{m,n}$ and its line graph $L(L_{m,n})$.

Definition 3.1. [16]

The lollipop graph $L_{m,n}$ is a graph obtained from a complete graph K_m and a path P_n, by joining one of the end vertices of P_n to one of the vertices of K_m. A lollipop graph $L_{4,3}$ is illustrated in Figure 1.

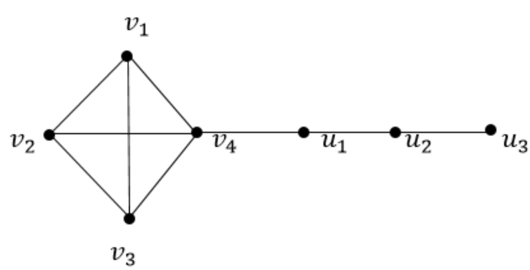

Figure 1: Lollipop graph $L_{4,3}$

Theorem 3.1. If $L_{m,n}$ is a lollipop graph with $m + n$ vertices, then the closeness for the lollipop $L_{m,n}$ is

$$C(L_{m,n}) = (m-1)\left[\frac{2^{n+1}-1}{2^n} + (m-2)\frac{1}{2}\right] + \frac{2^n(n-1)+1}{2^{n-1}}.$$

Proof. Due to form of lollipop graph, we can split into three subforms of the graph such as $C(v_i)$, where v_i is any vertex of K_m for $i = \overline{1, m-1}$, $C(P_{n+1} \sim K_{m-1})$, $C(P_{n+1})$. Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices that make up the complete graph K_m and $u_1, u_2, u_3, \ldots, u_n$ be the vertices that make up the path P_n. The lollipop graph $L_{m,n}$ contains a path P_{n+1} with the vertices $v_m, u_1, u_2, \ldots, u_n$. Any vertex v_i, $i = \overline{1, m-1}$ in K_m is at distance 1 to the other vertices of K_m, is at distance 2 to

u_1, is at distance 3 to u_2,..., is at distance $n+1$ to u_n. Hence, the closeness value of any vertex of v_i, $i=\overline{1,m-1}$ is

$$C(v_i)=(m-1)\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n+1}}$$

$$=(m-2)\frac{1}{2}+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n+1}}$$

We denote the closeness value of the vertices in P_{n+1} to the complete graph K_{m-1} by $C(P_{n+1}{\sim}K_{m-1})$. Then, distance between the vertex v_m and all vertices of K_{m-1} and u_1 is 1. Distance between the v_m and $u_2,\dots,u_n$ is $\frac{1}{2^2}$, $\frac{1}{2^3}$,..., $\frac{1}{2^{n+1}}$ as well , respectively. Hence,

$$C(P_{n+1}{\sim}K_{m-1})+C(v_i)=(\mathrm{m}-1)[\,(\mathrm{m}-2)\frac{1}{2}+2(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n+1}})]$$

We know from the Theorem 2.1 (c) that $C(P_{n+1})=2(n+1)-4+\frac{1}{2^{n-1}}=2(n-1)+\frac{1}{2^{n-1}}=\frac{2^n(n-1)+1}{2^{n-1}}$. Since $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n+1}}=\frac{2^{n+1}-1}{2^{n+1}}$, the closeness value of $L_{m,n}$ is

$$C\left(L_{m,n}\right)=(m-1)\left[(m-2)\frac{1}{2}+2\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n+1}}\right)\right]+C(P_{n+1})$$

$$=2(m-1)\frac{2^{n+1}-1}{2^{n+1}}+(m-1)(m-2)\frac{1}{2}+\frac{2^n(n-1)+1}{2^{n-1}}$$

$$=(m-1)\left[\frac{2^{n+1}-1}{2^n}+(m-2)\frac{1}{2}\right]+\frac{2^n(n-1)+1}{2^{n-1}}$$

The proof is completed.

Definition 3.2. [4]
Given a nonempty graph G, we define the line graph $L(G)$ of G as that graph whose vertices can be put in one to one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. It is relatively easy to determine the number of vertices and the number of edges of the line graph $L(G)$ of a graph G in terms of easily computed quantities in G. Indeed, if G is a (p,q) graph with degree sequence $d_1,d_2,\dots,d_p$ and $L(G)$ is a (p',q') graph, then $p'=q$ and $q'=\sum_{i=1}^{p}\binom{d_i}{2}$ since each edge of $L(G)$ corresponds to a pair of adjacent edges of G. Line graph $L(L_{4,3})$ is illustrated in Figure 2.

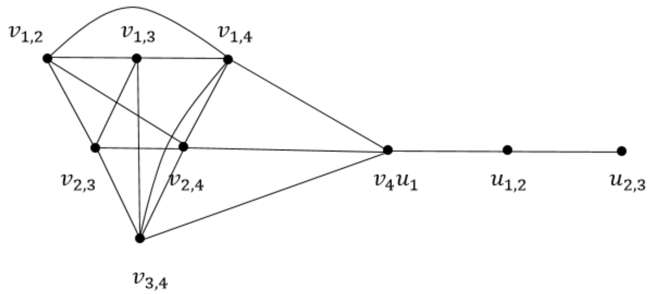

Figure 2: Line graph $L(L_{4,3})$

Theorem 3.2. If $L(L_{m,n})$ is a line graph of the lollipop graph $L_{m,n}$, then the closeness value of $L(L_{m,n})$ with $\binom{m}{2}+n=\frac{m(m-1)}{2}+n$ vertices is

$$C\left(L(L_{m,n})\right)=\frac{(m^3-3m^2+2m)(m+5)}{16}+2n-4+\frac{1}{2^{n-2}}+\frac{(m-1)(m+2)(2^n-1)}{2^{n+2}}$$

Proof. Due to form of $L(L_{m,n})$ graph, we can split into three subforms of $L_{m,n}$ in terms of closeness value such as $C(L(K_m))$, $C(P_n)$ and $C(L(K_m)\sim P_n)$. Let $v_{1,2},v_{1,3},\dots,v_{1,n},v_{2,3},v_{2,4},\dots,v_{2,n},v_{3,4},v_{3,5},\dots,v_{3,n},\dots,v_{m-1,n}$ are the vertices make up the graph $L(K_m)$ means $V\left(L(K_m)\right)=\{v_{i,j}\}$, where $i=\overline{1,m-1}$ and $j=\overline{\iota+1,m}$ and $|\,V\left(L(K_m)\right)|=\binom{m}{2}=\frac{m(m-1)}{2}$. $L(L_{m,n})$ contains a path graph P_n of order n with the vertices $v_mu_1,u_{1,2},u_{2,3},u_{3,4},\dots,u_{n-1,n}$. Every vertices of $L(K_m)$ are adjacent to $\left|V\left(L(K_{m-i})\right)\right|+2i$ vertices. So, distance between any vertex $v_{i,j}$ in $L(K_m)$ and $2m-4$ vertices in $L(K_m)$ is 1, for $m\geq 4$. Similarly, distance between any vertex $v_{i,j}$ in $L(K_m)$ and $\binom{m}{2}-2m+3$ remaining vertices in $L(K_m)$ is 2. Hence,

$$C\left(v_{i,j}\right)=(2m-4)\frac{1}{2}+\left(\frac{m(m-1)}{2}-2m+3\right)\frac{1}{2^2}$$

$$=(m-2)+\frac{m^2-5m+6}{2^3}$$
$$=(m-2)(1+\frac{m-3}{2^3})$$

Since we have $\binom{m}{2}$ vertices in $L(K_m)$,

$$C(L(K_m)) = \frac{m(m-1)}{2}(m-2)(1+\frac{m-3}{2^3})$$
$$= \frac{(m^3-3m^2+2m)(2^3+(m-3))}{2^4}$$
$$= \frac{(m^3-3m^2+2m)(m+5)}{16} \qquad (1)$$

We know from the Theorem 2.1 (c) $C(P_n) = 2n - 4 + \frac{1}{2^{n-2}}$ (2)
The vertex $v_m u_1$ is at distance 1 to $m-1$ vertices of $L(K_m)$ and is at distance 2 to $\frac{m(m-1)}{2} - (m-1)$ vertices of $L(K_m)$. So,

$$C(v_m u_1) = \frac{(m^2-3m+2)}{2}\frac{1}{2^2} + (m-1)\frac{1}{2}$$
$$= \frac{m-1}{2}(\frac{m-2}{2^2}+1)$$
$$= A$$

Closeness value of $u_{1,2}, u_{2,3}, u_{3,4}, \dots, u_{n-1,n}$ to all vertices of $L(K_m)$ is $\frac{1}{2}A$, $\frac{1}{2^2}A, \frac{1}{2^3}\ A, \dots, \frac{1}{2^{n-1}}A$, respectively. We will consider closeness value to the vertices in $L(K_m)$ and P_n in both direction. Hence, we have

$$C(L(K_m)\sim P_n) = 2(A+\frac{1}{2}A + \frac{1}{2^2}A, \frac{1}{2^3}\ A, \dots, \frac{1}{2^{n-1}}A)$$
$$= 2A(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}})$$
$$= 2(\frac{m-1}{2})(1+\frac{m-2}{2^2})\frac{2^n-1}{2^{n-1}}$$
$$= \frac{(m-1)(m+2)(2^n-1)}{2^{n+2}} \qquad (3)$$

By summing (1), (2) and (3), we have
$$C\left(L(L_{m,n})\right) = \frac{(m^3-3m^2+2m)(m+5)}{16} + 2n - 4 + \frac{1}{2^{n-2}} + \frac{(m-1)(m+2)(2^n-1)}{2^{n+2}}$$

The proof is completed.

4. Residual Closeness of Lollipop Graph and Line Graph

In order to evaluate residual closeness value, a vertex will be removed from the graph and the minimum closeness value will be calculated after removing. Therefore, the most sensitive vertex will be determined in the graph. In the lollipop graph structure, we will obtain four distinct value after removing. These motification can be get from removing the vertex u_1, the vertex v_m that is connected with u_1 , any vertex v_i, $i = \overline{1, m-1}$ except v_m in K_m and any vertex

u_i, $i = \overline{2,n}$ except u_1 in P_n. After determining the effect of these motifications on the graph in the next theorem, we will get residual closeness value of lollipop graph.

Theorem 4.1. Let $L_{m,n}$ be a lollipop graph with $m+n$ vertices. The residual closeness value of $L_{m,n}$ is

$$R\big(L_{m,n}\big) = \frac{(m-1)(m-2)}{2} + 2n - 4 + \frac{1}{2^{n-2}}$$

Proof. We have four different value after vertex removing.

Case 1. If u_1 is removed then we have two remaining graphs such as a complete graph K_m and a path graph P_{n-1}. Hence,

$$R_1 = C(K_m) + C(P_{n-1})$$

$$= \frac{m(m-1)}{2} + 2(n-1) - 4 + \frac{1}{2^{n-3}}$$

Case 2. If v_m is removed then we have two remaining graphs such as a complete graph K_{m-1} and a path graph P_n. Hence,

$$R_2 = C(K_{m-1}) + C(P_n)$$

$$= \frac{(m-1)(m-2)}{2} + 2n - 4 + \frac{1}{2^{n-2}}$$

Case 3. If any vertex v_i, $i = \overline{1, m-1}$ in the graph K_m is removed, then

$$R_3 = C(L_{m-1,n})$$
$$= (m-2)\left[\frac{2^{n+1}-1}{2^n} + (m-3)\frac{1}{2}\right] + \frac{2^n(n-1)+1}{2^{n-1}}$$

Case 4. If any vertex u_i, $i = \overline{2,n}$ in the graph P_n is removed, then

$$R_4 = C\big(L_{m,i-1}\big) + C(P_{n-i})$$
$$= (m-1)\left[\frac{2^i - 1}{2^{i-1}} + (m-2)\frac{1}{2}\right] + \frac{2^{i-1}(i-2)+1}{2^{i-2}} + 2(n-i) - 4 + \frac{1}{2^{n-i-2}}$$

If we compare Case 1, 2, 3 and 4, then it can be seen that the value comes from Case 2 is the minimum value. Hence,

$$R\big(L_{m,n}\big) = \frac{(m-1)(m-2)}{2} + 2n - 4 + \frac{1}{2^{n-2}}$$

The proof is completed.

Theorem 4.2. Let $L(L_{m,n})$ be a line graph of the lollipop graph $L_{m,n}$. Then residual closeness value of $L(L_{m,n})$ is

$$R\left(L\left(L_{m,n}\right)\right)=\frac{(m^3-3m^2+2m)(m+5)}{16}+2(n-1)-4+\frac{1}{2^{n-3}}$$

Proof. We will denote the vertices corresponding to the edges of the complete graph K_m in the lollipop graph $L_{m,n}$ by $v_{1,2},v_{1,3},v_{1,4},v_{1,5},\dots,v_{1,m},v_{2,3},v_{2,4},v_{2,5},\dots,v_{2,m},v_{3,4},v_{3,5},\dots,v_{3,m},\dots,v_{m-1,m}$ and the vertices corresponding to the edges of the path graph P_n in the lollipop graph $L_{m,n}$ by $v_m u_1, u_{1,2}, u_{2,3}, u_{3,4},\dots,u_{n-1,n}$. We have four different value after vertex removing. These values can be get from removing any vertex $v_{i,m}$, $i=\overline{1,m-1}$ in $L(K_m)$, $v_{i,j}$, $i=\overline{1,m-2}$, $j=\overline{\iota+1,m-1}$ in $L(K_m)$, the vertex $v_m u_1$ connecting $L(K_m)$ and $L(P_n)$, any vertex $u_{i,i+1}$, where $i=\overline{1,n-1}$ in $L(P_n)$

Case 1. If any vertex $v_{i,m}$, $i=\overline{1,m-1}$ in $L(K_m)$ is removed then the residual closeness value R_1 is obtained by subtracting the closeness value of this vertex in $L(K_m)$ from the $C(L(L_{m,n}))$. The closeness value of these vertices is $(m-2)(1+\frac{m-3}{2^3})$ in $L(K_m)$ and $1-\frac{1}{2^n}$ in $L(P_n)$. We must consider closeness value to the remaining vertices and these vertices in both direction. Hence,

$$R_1=C\left(L\left(L_{m,n}\right)\right)-2\{(m-2)\left(1+\frac{m-3}{2^3}\right)+(1-\frac{1}{2^n})\}$$

$$=\frac{(m^3-3m^2+2m)(m+5)}{16}+2n-4+\frac{1}{2^{n-2}}+\frac{(m-1)(m+2)(2^n-1)}{2^{n+2}}$$
$$-2\{(m-2)\left(1+\frac{m-3}{2^3}+1-\frac{1}{2^n}\right)\}$$

Case 2. If any vertex $v_{i,j}$, $i=\overline{1,m-2}$, $j=\overline{\iota+1,m-1}$ in $L(K_m)$ is removed then the residual closeness value R_2 is obtained by subtracting the closeness value of this vertex in $L(K_m)$ from the $C(L(L_{m,n}))$. The closeness value of these vertices is $(m-2)(1+\frac{m-3}{2^3})$ in $L(K_m)$ and $\frac{1}{2}(1-\frac{1}{2^n})$ in $L(P_n)$. We must consider closeness value to the remaining vertices and these vertices in both direction. Hence,

$$R_2=C\left(L\left(L_{m,n}\right)\right)-2\{(m-2)\left(1+\frac{m-3}{2^3}\right)+\frac{1}{2}\left(1-\frac{1}{2^n}\right)\}$$
$$=\frac{(m^3-3m^2+2m)(m+5)}{16}+2n-4+\frac{1}{2^{n-2}}+\frac{(m-1)(m+2)(2^n-1)}{2^{n+2}}-2(m-2)(1+\frac{m-3}{2^3})\text{-}(1-\frac{1}{2^n})$$

Case 3. If the vertex $v_m u_1$ in $L(P_n)$ is removed then the residual closeness value R_3 is

$$R_3=C\left(L(K_m)\right)+C(P_{n-1})$$
$$=\frac{(m^3-3m^2+2m)(m+5)}{16}+2(n-1)-4+\frac{1}{2^{n-3}}$$

Case 4. If the vertex $u_{i,i+1}$, where $i = \overline{1, n-1}$ in $L(P_n)$ is removed then the residual closeness value R_4 is

$$R_4 = C\big(L(L_{m,i}\big) + C(P_{n-1-i})$$
$$= \frac{(m^3 - 3m^2 + 2m)(m+5)}{16} + 2i - 4 + \frac{1}{2^{i-2}} + \frac{(m-1)(m+2)\big(2^i - 1\big)}{2^{i+2}} + 2(n-1-i) - 4 + \frac{1}{2^{n-3-i}}$$

If we compare Case 1, 2, 3 and 4, then it can be seen that the value comes from Case 3 is the minimum value. Hence,

$$R\left(L\big(L_{m,n}\big)\right) = \frac{(m^3 - 3m^2 + 2m)(m+5)}{16} + 2(n-1) - 4 + \frac{1}{2^{n-3}}$$

The proof is completed.

5. Conclusion

Stability and robustness of a network under some failures are defined as vulnerability. Closeness and vertex residual closeness are distinctive graph vulnerability parameters. Calculation of closeness and vertex residual closeness for simple graph types is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network. In this paper, closeness and vertex residual closeness formulas have been obtained for lollipop graph and its line graph.

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