Proceedings of IAM, V.12, N.2, 2023, pp.123-136
DOI: 10.30546/2225-0530.12.2.2023.123

MARCINKIEWICZ INTEGRAL AND ITS COMMUTATORS ON GENERALIZED ORLICZ-MORREY SPACES OF THE THIRD KIND

S.G. Hasanov[1], Nazlipinar Ali Serdar[2]

[1] Gandja State University, Gandja, Azerbaijan
[2]Department of Mathematics, Dumlupinar University, Kutahya, Turkey

e-mail: sabhasanov@gmail.com, ali.nazlipinar@dpu.edu.tr

Abstract: In this paper, we study the boundedness of the Marcinkiewicz operators μ_Ω and their commutators $[b,\mu_\Omega]$ on generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}$. We find the sufficient conditions on the pair (φ_1,φ_2) which ensure the boundedness of the operators μ_Ω and $[b,\mu_\Omega]$ from one generalized Orlicz-Morrey space M^{Φ,φ_1} to another M^{Φ,φ_2} .

Keywords: Generalized Orlicz-Morrey spaces, Marcinkiewicz integral, Commutator, BMO, Hardy operator.
AMS Subject Classification: 42B20, 42B25, 42B35.

1. Introduction and notations

Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [18,23]. The authors of [1,2] showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy-Littlewood operators, Calderon-Zygmund singular integral operators and fractional integral operators. A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces where the “Morrey-type measuring” of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one. Such spaces were first introduced and studied by Nakai [20]. Then another kind of generalized Orlicz-Morrey spaces were introduced by Sawano et al. [25]. Generalized Orlicz-Morrey spaces as the one introduced by Guliyev et al. [4], see also [8,10,11,13].

Let S^{n-1} be the unit sphere in R^n, $(n\geq 2)$ equipped with normalized Lebesgue measure dσ and $B(x,r)=\{y\in R^n : |x-y|<r\}$ be the open ball centered at x and radius r. Suppose $\Omega\in L^q(S^{n-1})$ with $1<q\leq\infty$ is homogeneous of degree zero and satisfies the cancelation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Marcinkiewicz operator μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty \left| F_{\Omega,t}(x) \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Let b be a locally integrable function on R^n, the commutator of b and μ_Ω is defined as follows

$$[b, \mu_\Omega] f(x) = \left(\int_0^\infty \left| F^b_{\Omega,t}(x) \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where

$$F^b_{\Omega,t}(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)dy.$$

It is well known that Marcinkiewicz operator plays an important role in harmonic analysis. Benedek et al. [3] proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L^p(R^n)$ for $1 < p < \infty$. The corresponding commutator $[b, \mu_\Omega]$ was first considered by Torchinsky and Wang in [26]. In 2002, Ding et al. [5] showed that if $\Omega \in L^q(S^{n-1})$, $q > 1$, then μ_Ω is bounded on $L^p(R^n)$ for $1 < p < \infty$.

In this paper, we consider the case when ϕ is dependents also on x. It is given a function $\phi : R^n \times (0,\infty) \to (0,\infty)$ as well as the Young function $\Phi : [0,\infty) \to [0,\infty)$. Denote by G_ϕ the set of all functions $\phi : R^n \times (0,\infty) \to (0,\infty)$ such that $\phi(x,t) \leq \phi(x,s)$ for all $t > s > 0$ and that $t \mapsto \Phi^{-1}(x, t^{-n})\phi(t)^{-1}$ is almost decreasing, that is, there exists a constant $C > 0$ independent of x such that $\Phi^{-1}(x, t^{-n})\phi(t)^{-1} \leq C\Phi^{-1}(x, s^{-n})\phi(s)^{-1}$

for all $0 < s < t < \infty$. Here $\Phi^{-1}(\cdot)$ is the inverse of $\Phi(\cdot)$. Denote by Δ_2 the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ such that the doubling condition:

$$\Phi(2t) \le C\Phi(t) \quad (t \ge 0) \tag{1}$$

holds for some constant $C \ge 2$, which is called doubling constant, and by ∇_2 the set of all convex functions $\Phi : [0,\infty) \to [0,\infty)$ such that the ∇_2-condition:

$$2C'\Phi(t) \le \Phi(2t) \quad (t \ge 0) \tag{2}$$

holds for some $C' > 1$. Note that C in (1) exceeds 2 when $\Phi \in \Delta_2 \cap \nabla_2$ due to (2). Recall also that the conjugate function Ψ of Φ is defined by:

$$\Psi(t) \equiv \sup\{st - \Phi(s) : s \ge 0\} \quad (t \ge 0).$$

Let Φ be a Young function. Recall that the Orlicz norm $\|f\|_{L^\Phi(E)}$ over a measurable set E in R^n is defined by:

$$\|f\|_{L^\Phi(E)} \equiv \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

Define $L^\Phi_{loc}(R^n)$ as the set of all measurable functions f for which $f \in L^\Phi(K)$ for all compact sets K in R^n.

We now define generalized Orlicz-Morrey spaces of the third kind. The *generalized Orlicz-Morrey space* $M^{\Phi,\phi}(R^n)$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}} \equiv \sup_{x\in R^n, r>0} \frac{1}{\phi(x,t)} \Phi^{-1}\left(\frac{1}{|B(x,r)|}\right) \|f\|_{L^\Phi(B(x,r))}$$

is finite.

Note that $M^{\Phi,\phi}(R^n)$ covers many classical function spaces.

Example 1.1. Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(R^n)$, the classical Morrey space $M^p_q(R^n)$, the generalized Morrey space $M^{\Phi,p}(R^n)$ and the Orlicz space $L^\Phi(R^n)$ with norm coincidence:

1. If $\Phi(t)=t^{p}$ and $\phi(t)=t^{-\frac{n}{p}}$, then $M^{\Phi,\phi}(R^{n})=L^{p}(R^{n})$ with equivalent norms.

2. If $\Phi(t)=t^{q}$ and $\phi(t)=t^{-\frac{n}{p}}$, then $M^{\Phi,\phi}(R^{n})$, which is denoted by $M_{q}^{p}(R^{n})$, is the classical Morrey space.

3. If $\Phi(t)=t^{p}$, then $M^{\Phi,\phi}(R^{n})=M^{p,\phi}(R^{n})$ is the generalized Morrey space which were discussed in [7,9,12,15,17,19].

4. If $\phi(t)=\Phi^{-1}(t^{-n})$, then $M^{\Phi,\phi}(R^{n})=L^{\Phi}(R^{n})$, which is beyond the reach of generalized Orlicz-Morrey spaces of the second kind defined in [25] according to an example constructed in [6].

Other definitions of generalized Orlicz-Morrey spaces can be found in [20,21,22,25]; Therefore, our definition of generalized Orlicz-Morrey spaces here is named “third kind”.

Therefore, the purpose of this paper is mainly to study the boundedness of Marcinkiewicz operator and its commutators in generalized Orlicz-Morrey spaces of the third kind.

By A $\lesssim$ B we mean that A $\le$ CB with some positive constant C independent of appropriate quantities. If A $\lesssim$ B and B $\lesssim$ A, we write A $\approx$ B and say that A and B are equivalent.

2 Marcinkiewicz integral in generalized Orlicz-Morrey spaces

In this section, we study the boundedness of integral operators in generalized Orlicz-Morrey spaces.

The following result concerning the boundedness of Marcinkiewicz integral operator μ_{Ω} on L^{p} is known.

Theorem 2.1. [27] Suppose that $1<p,q<\infty$ and $\Omega\in L^{q}(S^{n-1})$. Then, there is a constant C independent of f such that

$$\|\mu_{\Omega}(f)\|_{L^{p}(R^{n})}\le C\|f\|_{L^{p}(R^{n})}.$$

The following interpolation result is from [14].

Lemma 2.1. Let T be a sublinear operator of weak type (p,p) for any $p\in(1,\infty)$. Then T is bounded on $L^{\Phi}(R^{n})$, where Φ is a Young function satisfying $\Phi\in\Delta_{2}\cap\nabla_{2}$.

As a consequence of Lemma 2.1 and Theorem 2.1, we get the following result.

Corollary 2.1. Let Φ be a Young function and $\Omega \in L^{\infty}(S^{n-1})$. If $\Phi \in \Delta_2 \cap \nabla_2$, then μ_Ω is bounded on $L^{\Phi}(R^n)$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(r) := \int_r^{\infty} g(s) w(s) ds, \ \ r \in (0,\infty),$$

where w is a weight.

The following theorem was proved in [8].

Theorem 2.2. Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(r)$ be bounded outside a neighborhood of the origin. The inequality

$$\sup_{r>0} v_2(r) H_w^* g(r) \le C \sup_{r>0} v_1(r) g(r) \tag{3}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^{\infty} \frac{w(t)dt}{\sup\limits_{t<s<\infty} v_1(s)} < \infty .$$

Moreover, the value $C = B$ is the best constant for (3).

We also use the following lemma to prove our main estimates.

Lemma 2.2. For a Young function Φ and all balls B, the following inequality is valid

$$\|f\|_{L^1(B)} \le 2|B| \Phi^{-1}\left(|B|^{-1}\right) \|f\|_{L^{\Phi}(B)}.$$

Proof. The following analogue of the Holder inequality is known.

$$\left| \int_{R^n} f(x) g(x) dx \right| \le 2 \|f\|_{L^{\Phi}} \|g\|_{L^{\Phi}} . \tag{4}$$

For the proof of (4), see, for example [24].

The proof follows from Holder's inequality and the well known facts

$$r \le \Phi^{-1}(r) \widetilde{\Phi}^{-1}(r) \le 2r , \quad r > 0, \tag{5}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty , & r = \infty. \end{cases}$$

and $\|\chi_B\|_{L^\Phi} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}$.

Therefore, we have the following theorem

Theorem 2.3. Let Φ any Young function, φ_1, φ_2 and Φ satisfy the condition

$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\left(|B(x,s)|^{-1}\right)} \right) \Phi^{-1}\left(|B(x,s)|^{-1}\right) \frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Let also $\Omega \in L^\infty(S^{n-1})$. If Φ satisfy the condition $\Phi \in \Delta_2 \cap \nabla_2$, then the operator μ_Ω is bounded from $M^{\Phi,\varphi_1}(R^n)$ to $M^{\Phi,\varphi_2}(R^n)$.

Proof. For any ball $B = B(x_0, r)$, function $f(x)$ can be divided into two parts: $f = f\chi_{2B} + f\chi_{R^n \setminus 2B} := f_1 + f_2$, thus we have

$$\|\mu_\Omega f\|_{L^\Phi(B)} \le \|\mu_\Omega f_1\|_{L^\Phi(B)} + \|\mu_\Omega f_2\|_{L^\Phi(B)} \equiv I_1 + I_2.$$

For I_1, by $L^\Phi(R^n)$ boundedness of μ_Ω (see Corollary 2.1), we have

$$I_1 \le C\|f_1\|_{L^\Phi(R^n)} = \|f\|_{L^\Phi(2B)}.$$

From (5) we get

$$\Phi^{-1}\left(|B|^{-1}\right) \approx \Phi^{-1}\left(|B|^{-1}\right) r^n \int_{2r}^\infty \frac{dt}{t^{n+1}} \le C \int_{2r}^\infty \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}$$

and then

$$I_1 \le C\|f\|_{L^\Phi(2B)} \le C \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}. \quad (6)$$

For I_2, we first estimate $\mu_\Omega f_2(x)$ for any $x \in B$, since $y \in R^n \setminus 2B$, we have the following inequality: $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$, therefore we obtain

$$|\mu_\Omega f_2(x)| \le \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_2(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{\frac{1}{2}} dy \le C\|\Omega\|_{L^\infty(S^{n-1})} \int_{R^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^n} dy$$

.

By Fubini's theorem we have

$$\int_{{}^{C}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \approx \int_{{}^{C}(2B)} |f(y)| dy \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$

$$= \int_{2r}^{\infty} \int_{2r \le |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \le C \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.$$

By Lemma 2.2, we get

$$\int_{{}^{C}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \le C \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}. \tag{7}$$

Moreover

$$\|\mu_{\Omega}(f_2)\|_{L^{\Phi}(B)} \le \frac{C}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}$$

is valid. Thus

$$\|\mu_{\Omega}(f)\|_{L^{\Phi}(B)} \le C\|f\|_{L^{\Phi}(2B)} + \frac{C}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}$$

and from (6) we have

$$\|\mu_{\Omega}(f)\|_{L^{\Phi}(B)} \le \frac{C}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \frac{dt}{t}. \tag{8}$$

By inequality (8) and Theorem 2.2 we have

$$\|\mu_{\Omega}(f)\|_{M^{\Phi,\varphi_2}(R^n)} \le C \sup_{x_0 \in R^n, r>0} \varphi_2(x_0,r)^{-1} \int_{r}^{\infty} \Phi^{-1}\left(|B(x_0,t)|^{-1}\right) \|f\|_{L^{\Phi}(B(x_0,t))} \frac{dt}{t}$$

$$\le C \sup_{x_0 \in R^n, r>0} \varphi_1(x_0,r)^{-1} \Phi^{-1}\left(|B(x_0,r)|^{-1}\right) \|f\|_{L^{\Phi}(B(x_0,r))} = \|f\|_{L^{\Phi}(B(x_0,r))}.$$

Corollary 2.2. Let $\Omega \in L^{\infty}(S^{n-1})$, Φ be a Young function, $\varphi_1 \in G_{\Phi}$, and (φ_1, φ_2) satisfy the condition

$$\int_{r}^{\infty} \varphi_1(x,t) \frac{dt}{t} \le C \varphi_2(x,r),$$

where C does not depend on x and r. If Φ satisfy the condition $\Phi \in \Delta_2 \cap \nabla_2$, then the operator μ_{Ω} is bounded from $M^{\Phi,\varphi_1}(R^n)$ to $M^{\Phi,\varphi_2}(R^n)$.

3 Commutators of Marcinkiewicz integral in generalized Orlicz-Morrey spaces

In this section, we consider the commutators generalized by the singular integral operator, Marcinkiewicz operator and $BMO(R^n)$ function. A local integrable function $f \in L^{loc}(R^n)$, if it satisfies

$$\|b\|_* \equiv \sup_{x\in R^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $B(x,r)$ is ball centered at x and radius of r and $b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy$, then b belongs to $BMO(R^n)$, and $\|\cdot\|_*$ is the norm in $BMO(R^n)$. The following estimate is very convenient in applications.

Lemma 3.1. [16] Let $b \in BMO(R^n)$. Suppose $1 \le p < \infty$, $x \in R^n$ and $R > 2r > 0$, there exist constant $C > 0$, such that

$$\left| b_{B(x,R)} - b_{B(x,r)} \right| \le C \ln \frac{R}{r} \|b\|_*.$$

Before proving our theorems, we need the following lemma.

Lemma 3.2. [10] Let $b \in BMO(R^n)$ and Φ be a Young function with $\Phi \in \Delta_2$, then

$$\|b\|_* \approx \sup_{x\in R^n, r>0} \Phi^{-1}\left(|B(x,r)|^{-1}\right) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))}.$$

We will use the following statements on the boundedness of the weighted Hardy operator

$$\mathrm{H}_w^* g(r) := \int_r^\infty \left(1 + \ln \frac{s}{t}\right) g(s) w(s) ds, \quad r \in (0,\infty),$$

where w is a weight.

The following theorem is valid.

Theorem 3.1. Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\sup_{r>0} v_2(r) \mathrm{H}_w^* g(r) \le C \sup_{r>0} v_1(r) g(r) \tag{9}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \left(1+\ln\frac{s}{t}\right) \frac{w(t)dt}{\sup\limits_{t<s<\infty} v_1(s)} < \infty .$$

Moreover, the value $C = B$ is the best constant for (3.1).

Note that, Lemma 3.2 is proved analogously to [[8], Theorem 3.1].

The following result concerning the boundedness of commutators of Marcinkiewicz integral operator $[b, \mu_\Omega]$ on L^p is known.

Theorem 3.2. [27] Suppose that $1 < p, q < \infty$, $b \in BMO(R^n)$ and $\Omega \in L^q(S^{n-1})$. Then, there is a constant C independent of f such that

$$\|[b, \mu_\Omega](f)\|_{L^p(R^n)} \le C\|f\|_{L^p(R^n)}.$$

As a consequence of Lemma 2.1 and Theorem 3.2, we get the following result.

Corollary 3.1. Let Φ be a Young function, $b \in BMO(R^n)$ and $\Omega \in L^\infty(S^{n-1})$. If $\Phi \in \Delta_2 \cap \nabla_2$, then $[b, \mu_\Omega]$ is bounded on $L^\Phi(R^n)$.

Therefore, we get the following theorem

Theorem 3.3. Let $\Omega \in L^\infty(S^{n-1})$, $b \in BMO(R^n)$, Φ any Young function, φ_1, φ_2 and Φ satisfy the condition

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\left(|B(x,s)|^{-1}\right)}\right) \Phi^{-1}\left(|B(x,s)|^{-1}\right)\frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. If Φ satisfy the condition $\Phi \in \Delta_2 \cap \nabla_2$, then the operator $[b, \mu_\Omega]$ is bounded from $M^{\Phi,\varphi_1}(R^n)$ to $M^{\Phi,\varphi_2}(R^n)$.

Proof. For any ball $B = B(x_0, r)$, function $f(x)$ can be divided into two parts: $f = f\chi_{2B} + f\chi_{R^n\setminus 2B} := f_1 + f_2$, thus, we have

$$\|[b, \mu_\Omega]f\|_{L^\Phi(B)} \le \|[b, \mu_\Omega]f_1\|_{L^\Phi(B)} + \|[b, \mu_\Omega]f_2\|_{L^\Phi(B)} \equiv J_1 + J_2 .$$

For J_1, by $L^\Phi(R^n)$ boundedness of $[b, \mu_\Omega]$ (see Corollary 3.1), from (6) we have

$$J_1 \le C\|f_1\|_{L^\Phi(R^n)} = \|f\|_{L^\Phi(2B)} \le C\frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}\int_{2r}^\infty \|f\|_{L^\Phi(B(x_0,t))}\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t}$$

.

For J_2, observe that for any $x \in B$, since $y \in R^n \setminus 2B$, it has the following inequality: $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$, therefore we obtain

$$|[b,\mu_\Omega]f_2(x)| \le C \int_{R^n\setminus 2B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(x)-b(y)|\,|f(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{\frac{1}{2}} dy$$

$$\le C \int_{R^n\setminus 2B} \frac{|\Omega(x-y)|}{|x-y|^{n}} |b(x)-b(y)|\,|f(y)|\, dy$$

$$\le C \,\|\Omega\|_{L^\infty(S^{n-1})} \int_{R^n\setminus 2B} \frac{|b(x)-b(y)|}{|x_0-y|^{n}} \,|f(y)|\, dy .$$

Then

$$\|[b,\mu_\Omega]f_2\|_{L^\Phi(B)} \le C \left\| \int_{R^n\setminus 2B} \frac{|b(y)-b(\cdot)|}{|x_0-y|^{n}} \,|f(y)| dy \right\|_{L^\Phi(B)}$$

$$\le C \left\| \int_{R^n\setminus 2B} \frac{|b(y)-b_B|}{|x_0-y|^{n}} \,|f(y)| dy \right\|_{L^\Phi(B)} + \left\| \int_{R^n\setminus 2B} \frac{|b(\cdot)-b_B|}{|x_0-y|^{n}} \,|f(y)| dy \right\|_{L^\Phi(B)}$$

$$= J_{21} + J_{22}.$$

For J_1 we have

$$J_{21} \approx \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{R^n\setminus 2B} \frac{|b(y)-b_B|}{|x_0-y|^{n}} |f(y)|\, dy$$

$$\approx \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{R^n\setminus 2B} |b(y)-b_B|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$

$$= \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \int_{2r \le |x_0-y| < t} |b(y)-b_B|\,|f(y)|\, dy \frac{dt}{t^{n+1}}$$

$$= \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_B|\,|f(y)|\, dy \frac{dt}{t^{n+1}} .$$

Hence

$$J_{21} \le C\,\frac{1}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b(y) - b_{B(x_0,t)}\right| |f(y)|\,dy\,\frac{dt}{t^{n+1}}$$

$$= \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \int_{B(x_0,t)} |f(y)|\,dy\,\frac{dt}{t^{n+1}} \;.$$

Applying Holder's inequality, by (5) and Lemmas 2.2, 3.1 and 3.2 we get

$$J_{21} \le C\,\frac{1}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left\|b(\cdot) - b_{B(x_0,t)}\right\|_{L^{\tilde{\Phi}}(B(x_0,t))} \|f\|_{L^{\Phi}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$

$$+ \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \|f\|_{L^{\Phi}(B(x_0,t))}\,\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t}$$

$$\le C\,\frac{\|b\|_*}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left(1 + \ln\frac{s}{t}\right) \|f\|_{L^{\Phi}(B(x_0,t))}\,\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t} \;.$$

For J_{22} we obtain

$$J_{22} \approx \|b(\cdot) - b_B\|_{L^{\Phi}(B)} \int_{R^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^n}\,dy.$$

By Lemma 3.2 and the inequality (7), we get

$$J_{22} \le C\,\frac{\|b\|_*}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{R^n \setminus 2B} \frac{|f(y)|}{|x_0 - y|^n}\,dy$$

$$\le C\,\frac{\|b\|_*}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))}\,\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t}.$$

Combining the estimates for J_{21} and J_{22} we have

$$\left\|[b,\mu_\Omega] f_2\right\|_{L^{\Phi}(B)} \le C\,\frac{\|b\|_*}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left(1 + \ln\frac{s}{t}\right) \|f\|_{L^{\Phi}(B(x_0,t))}\,\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t}. \tag{10}$$

Again combining the estimates for $[b,\mu_\Omega] f_1$ and $[b,\mu_\Omega] f_2$ we have

$$\left\|[b,\mu_\Omega] f\right\|_{L^{\Phi}(B)} \le C\,\frac{\|b\|_*}{\Phi^{-1}\left(|B|^{-1}\right)} \int_{2r}^{\infty} \left(1 + \ln\frac{s}{t}\right) \|f\|_{L^{\Phi}(B(x_0,t))}\,\Phi^{-1}\left(|B(x_0,t)|^{-1}\right)\frac{dt}{t}$$

By inequality (10) and Theorem 2.2 we have

$$\left\|[b,\mu_\Omega](f)\right\|_{M^{\Phi,\varphi_2}(R^n)} \le C \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r)^{-1}\int_r^\infty \Phi^{-1}\left(\left|B(x_0,t)\right|^{-1}\right)\|f\|_{L^\Phi(B(x_0,t))}\frac{dt}{t}$$

$$\le C \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r)^{-1}\Phi^{-1}\left(\left|B(x_0,t)\right|^{-1}\right)\|f\|_{L^\Phi(B(x_0,t))} = \|f\|_{M^{\Phi,\varphi_1}(R^n)}.$$

Corollary 3.2. Let $\Omega\in L^\infty\left(S^{n-1}\right)$, $b\in BMO(R^n)$, Φ any Young function, $\varphi_1\in G_\Phi$ and (φ_1,φ_2) satisfy the condition

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)\varphi_1(x,t)\frac{dt}{t}\le C\varphi_2(x,r),$$

where C does not depend on x and r. If Φ satisfy the condition $\Phi\in\Delta_2\cap\nabla_2$, then the operator $[b,\mu_\Omega]$ is bounded from $M^{\Phi,\varphi_1}(R^n)$ to $M^{\Phi,\varphi_2}(R^n)$.

Acknowledgements. The author would like to express his gratitude to the referees for his (her) very valuable comments and suggestions.

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