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THE INVERSE PROBLEM OF DISPLACEMENT OF THE 6R MANIPULATOR WITH A SEQUENTIAL STRUCTURE BY MEANS OF DOUBLE QUATERNIONS

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Abstract. In this research work, the solution of forward and inverse problems of displacements for the 6R manipulator with a serial structure is investigated. The algebraic method of dual quaternions is employed as the operator of spatial transformations. Furthermore, due to the introduction of intermediate angles which are linear combinations of real angles of rotation in revolute kinematic pairs, the contour equations are greatly simplified, and this makes it possible to express the equations in an implicit form. The total number of these equations is 16: the first six equations are linear with respect to the unknown sins and cosines of intermediate angles; the following eight equations express obvious trigonometric dependences - equality to the unit of the sum of squares of unknowns; the last two nonlinear equations express additional connections between unknowns.

Keywords: quaternion, dual, space, transformation, intermediate angle, manipulator.

AMS Subject Classification: 11R52, 15A04, 20M20

1. Introduction.

Several numbers of research work about the kinematical study of both serial and platform type manipulators are explored. In the article [8], using dual quaternions as the operator of the spatial operator and the orientation of a rigid body, the equations of displacement of a manipulator with an open chain 6R are compiled. In [7], quaternions are applied to the kinematic analysis of complex spatial mechanisms, brief remarks are made on the Kotelnikov transfer principle. By introducing intermediate angles, the contour equations are greatly simplified. Collins, C. L., and McCarthy, J. M studied workspace and singular configurations on a parallel structure RPR manipulator where they used a quaternion mathematical apparatus [3]. In the article [9] Martines et al. presented quaternion operators to describe orientation, angular velocities, and accelerations in the spherical motion of a rigid body. Dai, S. J gave a historical overview of the development of the theory of rigid body displacement to determine the final rotation by means of Rodriguez parameters [4]. In [10] Pennestri et.al. examined the use of dual quaternions as a tool for determining rigid body movements using biomechanics. In the article [1] Banavar et.al. the synthesis and analysis of a spherical robot using quaternion algebra is presented. Liao, Q used dual quaternions to solve the inverse problem of manipulators of the 6R type [6]. In the article [11], Thomas F. showed that dual quaternions provide an efficient and concise way to combine the translational and rotational motion of a rigid body in space in one mathematical apparatus. The use of double quaternions is seen as a breakthrough in modern robotics theory.In paper [14] Zhou W, et al. propose a new kinematic analysis algorithm for the Stewart platform with six degrees of freedom based on the dual quaternion. The forward kinematic algorithms for the 6-6R and 6-2RP3R manipulators can be expressed as a unified mathematical model. In the article [13] by the authors XiaoLong Yang et al., the main task is to develop an effective method for solving the direct problem of the kinematics of robots of parallel structures with six degrees of freedom. An efficient algorithm for deriving equations and their solution is proposed. The new algorithm was compared with the solution using Newton's method, resulting in a time cost of 0.2187 ms and 14.25 ms, respectively. According to the authors, the examples demonstrate the effectiveness of the proposed method. Chelnokov's article [2] considers the problem of bringing a coordinate system associated with a body to a reference system moving relative to a fixed coordinate system with a given instantaneous speed of the propeller. Biquaternion kinematic equations of motion of a rigid body using normalized and non-normalized dual quaternions are used as a mathematical model, and the projections of the screw of the instantaneous velocity of the body on the coordinate axes associated with the body are used as control parameters. The constructed theory of rigid body motion control is used to solve inverse problems of manipulator kinematics. Wang JY, et al. in paper [12] consider the problem of coordinated control of translational motion and rotary motion between two spacecraft. Using a dual quaternion, a dynamic model has been developed in which the relationship between translational and rotational motion is indicated. Theoretically proved the convergence of a closed system in the presence of external disturbances. The validity of the proposed approach is demonstrated by numerical simulation. In [5] Jing Li, to describe the rotational and most general motion of a rigid body, introduces the vector and screw of the rigid body's motion, respectively, and thus updates the quaternion and the dual quaternion, respectively; then, information about the relative position of the leading and driven rigid body is displayed based on the developed algorithm of helical motion. The simulation results show that the proposed method eliminates both the disadvantages associated with the separate consideration of orientation and translational motion when using the traditional algorithm, but also has a higher accuracy than the traditional algorithm.

In this research paper, we solve the forward and inverse problems of the 6R manipulator by using the dual quaternions as the operator of the most general spatial transformation, as well as introducing intermediate angles.

2. Determination of the positions of open kinematic chain.

It is known that investigations of displacements of manipulators are determined by two main methods:

- forward problem: displacements in kinematic pairs, the position and orientation of the gripper and the associated coordinate system are determined;
- inverse problem: determination of displacements in kinematic pairs of an open chain, providing the required position and orientation of a rigid body in space.

Let's look at these problems on a serial type 6R manipulator (Fig. 1).

2.1. Forward problem solution.

The dual quaternion specifying the orientation and displacement of the coordinate system **(** $\bar{\boldsymbol{i}}_{\boldsymbol{1}}, \bar{\boldsymbol{i}}_{\boldsymbol{2}}, \bar{\boldsymbol{i}}_{\boldsymbol{3}}$**)** associated with the gripper relative to the original coordinate system **(** $\bar{\boldsymbol{I}}_{\boldsymbol{1}}, \bar{\boldsymbol{I}}_{\boldsymbol{2}}, \bar{\boldsymbol{I}}_{\boldsymbol{3}}$ **)** denoted by $\boldsymbol{X}$:

$$\boldsymbol{X} = X_o + X_1\bar{\boldsymbol{i}}_{\boldsymbol{1}} + X_2\bar{\boldsymbol{i}}_{\boldsymbol{2}} + X_3\bar{\boldsymbol{i}}_{\boldsymbol{3}}, \tag{1}$$

It can be expressed by the following quaternion product:

$$\boldsymbol{X} = \boldsymbol{\Lambda_1}\circ\boldsymbol{A_1}\circ\boldsymbol{\Lambda_2}\circ\boldsymbol{A_2}\circ, \dots ,\circ\boldsymbol{\Lambda_6}\circ\boldsymbol{A_6}, \tag{2}$$

where,

$\boldsymbol{\Lambda_i} = cos\Phi_i + \bar{\boldsymbol{i}}_{\boldsymbol{3}} sin\,\Phi_i\,, (i = 1,2, \dots ,6)$ dual quaternions characterizing displacements (rotations) in kinematic pairs,

$\Phi_i = \varphi_i + \delta\varphi_i^0$, variable dual angles, φ_i - half values of the relative angles of rotation of links in rotational kinematic pairs, φ_i^0- half constant distances in rotational kinematic pairs (Fig. 1);

$\boldsymbol{A_j} = cosB_j + \bar{\boldsymbol{i}}_{\boldsymbol{2}} sin\,B_j\,, (j = 1,2, \dots ,6)$dual quaternions characterizing the geometry of the manipulator links,

$B_j = \beta_j + \delta\beta_j^O$, constant dual angles, β_j - half values of constant angles between adjacent link axes, β_j^O- half values of constant shortest distances between adjacent axes.

Taking these remarks into account, quaternion multiplication (2) can also be expressed in the following form:

$$\boldsymbol{X} = \left(cos\Phi_1 + \bar{\boldsymbol{i}}_{\boldsymbol{3}} sin\,\Phi_1\right)\circ (cosB_1 + \bar{\boldsymbol{i}}_{\boldsymbol{2}} sin\,B_1)\circ \left(cos\Phi_2 + \bar{\boldsymbol{i}}_{\boldsymbol{3}} sin\,\Phi_2\right)\circ(cosB_2 + \bar{\boldsymbol{i}}_{\boldsymbol{2}} sin\,B_2)\circ \cdots\circ\left(cos\Phi_6 + \bar{\boldsymbol{i}}_{\boldsymbol{3}} sin\,\Phi_6\right)\circ (cosB_6 + \bar{\boldsymbol{i}}_{\boldsymbol{2}} sin\,B_6),$$

Thus, the direct task of determining the displacements of the manipulator is reduced to performing quaternion multiplication and therefore is not difficult from a mathematical point of view. After performing quaternion multiplication and equating the coefficients at the units of *1*, $\bar{\boldsymbol{i}}_{\boldsymbol{1}}, \bar{\boldsymbol{i}}_{\boldsymbol{2}}, \bar{\boldsymbol{i}}_{\boldsymbol{3}}$, we determine the components of the dual quaternion $\boldsymbol{X}$, which specifies the position and orientation of the rigid body in space.

3. Solution of the inverse problem.

Expressing the dual quaternion X defining the orientation of the coordinate system $(\bar{\boldsymbol{i}}_1, \bar{\boldsymbol{i}}_2, \bar{\boldsymbol{i}}_3)$ associated with the gripper relative to the original coordinate system $(\bar{\boldsymbol{I}}_1, \bar{\boldsymbol{I}}_2, \bar{\boldsymbol{I}}_3)$ in the following way:

$$\boldsymbol{X} = X_o + X_1\bar{\boldsymbol{i}}_1 + X_2\bar{\boldsymbol{i}}_2 + X_3\bar{\boldsymbol{i}}_3 =$$
$$= (x_o + \delta x_0^0) + \bar{\boldsymbol{i}}_1(x_1 + \delta x_1^0) + \bar{\boldsymbol{i}}_2(x_2 + \delta x_2^0) + \bar{\boldsymbol{i}}_3(x_3 + \delta x_3^0)$$

Let us define displacements in kinematic pairs that provide a given position of a rigid body. We transform expression (2) to the following form:

$$\boldsymbol{\Lambda_1}\circ\boldsymbol{A_1}\circ\boldsymbol{\Lambda_2}\circ\boldsymbol{A_2}\circ\boldsymbol{\Lambda_3}\circ\boldsymbol{A_3} = \boldsymbol{X}\circ\widetilde{\boldsymbol{A}}_6\circ\widetilde{\boldsymbol{\Lambda}}_6\circ\widetilde{\boldsymbol{A}}_5\circ\widetilde{\boldsymbol{\Lambda}}_5\circ\widetilde{\boldsymbol{A}}_4\circ\widetilde{\boldsymbol{\Lambda}}_4 \qquad (3)$$

Where $\widetilde{\boldsymbol{A}}_6$, $\widetilde{\boldsymbol{\Lambda}}_6$, $\widetilde{\boldsymbol{A}}_5$, $\widetilde{\boldsymbol{\Lambda}}_5$, $\widetilde{\boldsymbol{A}}_4$, $\widetilde{\boldsymbol{\Lambda}}_4$ are dual quaternions, conjugate to dual quaternions $\boldsymbol{A_6}$, $\boldsymbol{\Lambda_6}$, $\boldsymbol{A_5}$, $\boldsymbol{\Lambda_5}$, $\boldsymbol{A_4}$, $\boldsymbol{\Lambda_4}$ respectively. Quaternion products in expression (3) will be denoted by the following dual quaternions:

$$\boldsymbol{\Lambda_1}\circ\boldsymbol{A_1}\circ\boldsymbol{\Lambda_2}\circ\boldsymbol{A_2}\circ\boldsymbol{\Lambda_3}\circ\boldsymbol{A_3} = \boldsymbol{M}; \quad \widetilde{\boldsymbol{A}}_6\circ\widetilde{\boldsymbol{\Lambda}}_6\circ\widetilde{\boldsymbol{A}}_5\circ\widetilde{\boldsymbol{\Lambda}}_5\circ\widetilde{\boldsymbol{A}}_4\circ\widetilde{\boldsymbol{\Lambda}}_4 = \boldsymbol{N} \qquad (4)$$

Taking into account (4), expression (3) takes the following form:

$$\boldsymbol{M} = \boldsymbol{X}\circ\boldsymbol{N} \qquad (5)$$

We also can express the dual quaternions $\boldsymbol{M}$ and $\boldsymbol{N}$ by the components:
$\boldsymbol{M} = M_o + M_1\bar{\boldsymbol{i}}_1 + M_2\bar{\boldsymbol{i}}_2 + M_3\bar{\boldsymbol{i}}_3$; $\boldsymbol{N} = N + N_1\bar{\boldsymbol{i}}_1 + N_2\bar{\boldsymbol{i}}_2 + N_3\bar{\boldsymbol{i}}_3$

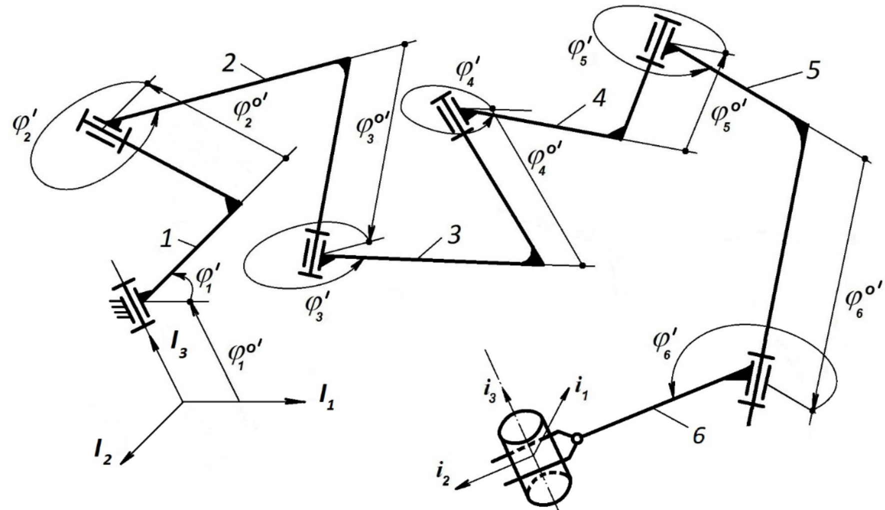

Figure. 1. Spatial 6R manipulator

While performing the quaternion multiplication of expressions (4), we use the intermediate angles described in [8]:

$$\Psi_1 = \Phi_2 + \Phi_3 + \Phi_4; \quad \Psi_5 = \Phi_5 + \Phi_6 + \Phi_7$$
$$\Psi_2 = \Phi_2 - \Phi_3 + \Phi_4; \quad \Psi_6 = \Phi_5 - \Phi_6 + \Phi_7$$
$$\Psi_3 = \Phi_2 + \Phi_3 - \Phi_4; \quad \Psi_7 = \Phi_5 + \Phi_6 - \Phi_7$$
$$\Psi_4 = \Phi_2 - \Phi_3 - \Phi_4; \quad \Psi_8 = \Phi_5 - \Phi_6 - \Phi_7,$$

There are two additional dependencies between these angles:

$$\Psi_1 + \Psi_4 = \Psi_2 + \Psi_3 \qquad (6)$$
$$\Psi_5 + \Psi_8 = \Psi_6 + \Psi_7$$

Thus, performing the quaternion multiplication of expressions (4) and equating the coefficients at the vectors *1*, $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$,, we obtain the components of the quaternions ***M u N:***

$$M_o = cosB_1cosB_2cosB_3\,cos(\Phi_1 + \Phi_2 + \Phi_3) - sinB_1sinB_2cosB_3\,cos(\Phi_1 - \Phi_2 + \Phi_3) -$$
$$-cosB_1sinB_2sinB_3\,cos(\Phi_1 + \Phi_2 - \Phi_3) - sinB_1cosB_2sinB_3\,cos(\Phi_1 - \Phi_2 - \Phi_3)\;;$$

$$M_1 = -cosB_1sinB_2cosB_3\,sin(\Phi_1 + \Phi_2 - \Phi_3) - sinB_1cosB_2cosB_3\,sin(\Phi_1 - \Phi_2 - \Phi_3) -$$
$$-cosB_1cosB_2sinB_3\,sin(\Phi_1 + \Phi_2 + \Phi_3) + sinB_1sinB_2sinB_3\,sin(\Phi_1 - \Phi_2 + \Phi_3)\;;$$

$M_2 = cosB_1 cosB_2 sinB_3 \cos(\Phi_1 + \Phi_2 + \Phi_3)$
$- sinB_1 sinB_2 sinB_3 \cos(\Phi_1 - \Phi_2 + \Phi_3) +$
$+cosB_1 sinB_2 cosB_3 \cos(\Phi_1 + \Phi_2 - \Phi_3) + sinB_1 cosB_2 cosB_3 \cos(\Phi_1 - \Phi_2 - \Phi_3);$
$M_3 = -cosB_1 sinB_2 sinB_3 \sin(\Phi_1 + \Phi_2 - \Phi_3)$
$- sinB_1 cosB_2 sinB_3 \sin(\Phi_1 - \Phi_2 - \Phi_3) +$
$+cosB_1 cosB_2 cosB_3 \sin(\Phi_1 + \Phi_2 + \Phi_3) - sinB_1 sinB_2 cosB_3 \sin(\Phi_1 - \Phi_2 + \Phi_3).$

$N_o = cosB_6 cosB_5 cosB_4 \cos(\Phi_4 + \Phi_5 + \Phi_6) - sinB_1 sinB_2 cosB_3 \cos(\Phi_4 - \Phi_5 + \Phi_6) -$
$-cosB_6 sinB_5 sinB_4 \cos(\Phi_4 + \Phi_5 - \Phi_6) - sinB_6 cosB_5 sinB_4 \cos(\Phi_4 - \Phi_5 - \Phi_6)$;
$N_1 = cosB_6 sinB_5 cosB_4 \sin(\Phi_4 - \Phi_5 + \Phi_6)$
$- sinB_6 cosB_5 cosB_4 \sin(\Phi_4 + \Phi_5 + \Phi_6) +$
$+cosB_6 cosB_5 sinB_4 \sin(\Phi_4 - \Phi_5 - \Phi_6) + sinB_6 sinB_5 sinB_4 \sin(\Phi_4 + \Phi_5 - \Phi_6)$;
$N_2 = cosB_6 cosB_5 sinB_4 \cos(\Phi_4 - \Phi_5 + \Phi_6) - sinB_6 sinB_5 sinB_4 \cos(\Phi_4 + \Phi_5 - \Phi_6) +$
$+cosB_6 sinB_5 cosB_4 \cos(\Phi_4 + \Phi_5 - \Phi_6) + sinB_6 cosB_5 cosB_4 \cos(\Phi_4 + \Phi_5 + \Phi_6)$
$N_3 = -cosB_6 sinB_5 sinB_4 \sin(\Phi_4 + \Phi_5 - \Phi_6) + sinB_6 cosB_5 sinB_4 \sin(\Phi_4 - \Phi_5 - \Phi_6) +$
$+cosB_6 cosB_5 cosB_4 \sin(\Phi_4 + \Phi_5 + \Phi_6) + sinB_6 sinB_5 cosB_4 \sin(\Phi_4 - \Phi_5 + \Phi_6).$

We will accept the following designations that define the geometry of the links of the mechanism:

$A111 = cosB_1 cosB_2 cosB_3$; $A112 = cosB_1 cosB_2 sinB_3$;
$A121 = cosB_1 sinB_2 cosB_3$; $A122 = cosB_1 sinB_2 sinB_3$;
$A211 = sinB_1 cosB_2 cosB_3$; $A212 = sinB_1 cosB_2 sinB_3$
$A221 = sinB_1 sinB_2 cosB_3$; $A222 = sinB_1 cosB_2 sinB_3$;
$B111 = cosB_6 cosB_5 cosB_4$; $B112 = cosB_6 cosB_5 sinB_4$;
$B121 = cosB_6 sinB_5 cosB_4$; $B122 = cosB_6 sinB_5 sinB_4$;
$B211 = sinB_6 cosB_5 cosB_4$; $B212 = sinB_6 cosB_5 sinB_4$;
$B221 = sinB_1 sinB_2 cosB_3$; $B222 = sinB_6 sinB_5 sinB_4$.

Substituting the components of the dual quaternions $\boldsymbol{X}$ and $\boldsymbol{N}$ into the quaternion multiplication (5), performing quaternion multiplication and equating the components at unit vectors *1*, $\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3}$ we obtain four dual expressions:

$M_o = X_o N_o - X_1 N_1 - X_2 N_2 - X_3 N_3$

$$M_1 = X_o N_1 + X_1 N_0 + X_2 N_3 - X_3 N_2 \quad (7)$$
$$M_2 = X_o N_2 - X_1 N_3 + X_2 N_0 - X_3 N_1$$
$$M_3 = X_o N_3 + X_1 N_2 - X_2 N_1 + X_3 N_0 \ ,$$

Taking into account the above accepted designations, we can write:

$$A111 \cdot X_1 - A221 \cdot X_3 - A122 \cdot X_5 - A212 \cdot X_7 =$$
$$= \{B111 \cdot X_9 - B221 \cdot X_{15} - B122 \cdot X_{11} - B212 \cdot X_{13}\} -$$
$$-\{-B121 \cdot X_{16} + B211 \cdot X_{10} - B112 \cdot X_{14} - B222 \cdot X_{12}\} -$$
$$-\{-B121 \cdot X_{15} - B211 \cdot X_9 - B112 \cdot X_{13} + B222 \cdot X_{11}\} -$$
$$-\{-B111 \cdot X_{10} - B221 \cdot X_{16} + B122 \cdot X_{12} - B212 \cdot X_{14}\}$$

$$-A121 \cdot X_6 - A211 \cdot X_8 - A112 \cdot X_2 + A222 \cdot X_4 =$$
$$\{-B121 \cdot X_{16} + B211 \cdot X_{10} - B112 \cdot X_{14} - B222 \cdot X_{12}\} +$$
$$+\{B111 \cdot X_9 - B221 \cdot X_{15} - B122 \cdot X_{11} - B212 \cdot X_{13}\} +$$
$$+\{-B111 \cdot X_{10} - B221 \cdot X_{16} + B122 \cdot X_{12} - B212 \cdot X_{14}\} -$$
$$-\{-B121 \cdot X_{15} - B211 \cdot X_9 - B112 \cdot X_{13} + B222 \cdot X_{11}\}$$

$$A112 \cdot X_1 - A222 \cdot X_3 + A121 \cdot X_5 + A211 \cdot X_7 =$$
$$\{-B121 \cdot X_{15} - B211 \cdot X_9 - B112 \cdot X_{13} + B222 \cdot X_{11}\} -$$
$$-\{-B111 \cdot X_{10} - B221 \cdot X_{16} + B122 \cdot X_{12} - B212 \cdot X_{14}\} + \quad (8)$$
$$+\{B111 \cdot X_9 - B221 \cdot X_{15} - B122 \cdot X_{11} - B212 \cdot X_{13}\} +$$
$$+\{-B121 \cdot X_{16} + B211 \cdot X_{10} - B112 \cdot X_{14} - B222 \cdot X_{12}\}$$

$$-A122 \cdot X_6 - A212 \cdot X_8 + A111 \cdot X_2 - A221 \cdot X_4 =$$
$$\{-B111 \cdot X_{10} - B221 \cdot X_{16} + B122 \cdot X_{12} - B212 \cdot X_{14}\} +$$
$$+\{-B121 \cdot X_{15} - B211 \cdot X_9 - B112 \cdot X_{13} + B222 \cdot X_{11}\} -$$
$$-\{-B121 \cdot X_{16} + B211 \cdot X_{10} - B112 \cdot X_{14} - B222 \cdot X_{12}\} +$$
$$+\{B111 \cdot X_9 - B221 \cdot X_{15} - B122 \cdot X_{11} - B212 \cdot X_{13}\}$$

where

$$Cos\Psi_1 = X_1; \ Sin\Psi_1 = X_2; \ Cos\,\Psi_2 = X_3; \ \ Sin\,\Psi_2 = X_3;$$
$$Cos\,\Psi_3 = X_5; Sin\,\Psi_3 = X_6; Cos\,\Psi_4 = X_7; \ \ Sin\,\Psi_4 = x_8;$$
$$Cos\Psi_5 = X_9; \ Sin\Psi_5 = X_{10}; \ Cos\Psi_6 = X_{11}; Sin\Psi_6 = X_{12};$$
$$Cos\Psi_7 = X_{13}; Sin\Psi_7 = X_{14}; Cos\Psi_8 = X_{15}; Sin\Psi_8 = X_{16};$$

There is one condition between the four dual equations, which reflects the equality to unity of the norm of the dual quaternion:

$$M_0^2 + M_1^2 + M_2^2 + M_3^2 = 1$$

Therefore, discarding one of any equations (8), we obtain three independent dual equations, which will be equivalent to six real equations. These equations are linear with respect to the unknowns X_i *(i=1,2,…16)*, which are sins and cosines of intermediate angles Ψ_j *(j=1,2,…,8)*.The following 8 equations represent the conditions for equality to the unit of the sum of squares:

$$x_1^2 + x_2^2 = 1;\ x_3^2 + x_4^2 = 1;\ x_5^2 + x_6^2 = 1;\ x_7^2 + x_8^2 = 1;$$
$$x_9^2 + x_{10}^2 = 1;\ x_{11}^2 + x_{12}^2 = 1;\ x_{13}^2 + x_{14}^2 = 1;\ x_{15}^2 + x_{16}^2 = 1$$

The last two equations express additional conditions (8).
Thus, to determine 16 unknowns, there are 16 equations, the first six of which are linear with respect to the unknowns x_i (*i=1,2,...,16*).After determining the intermediate angles Ψ_j *(j=1,2,…,8),* the calculation of the angles φ_k *(k=1,2,...,6)* is not difficult.

Conclusions

The use of dual quaternions as operators of the most general spatial transformation is effective not only in the kinematic analysis of spatial mechanisms with a closed kinematic chain, but also in the kinematic analysis of mechanisms with open chains. The introduction of intermediate angles makes it possible to significantly simplify the equations, as well as to implicitly express the dependence of the geometric and kinematic parameters of the 6R manipulator.

Reference

1. Banavar N.R. et. al, Design and analysis of a spherical mobile robot, Mech. Mach. Theory, Vol.45, (2010), pp.130-136.
2. Chelnokov I.N., Biquaternion solution of the kinematic control problem for the motion of a rigid body and its application to the solution of inverse problems of robotmanipulator kinematics, Mech Solid, Vol.48, (2013), pp.31–46.
3. Collins C.L., McCarthy J.M., The quartic singularity surfaces of planar platforms in the Clifford algebra of the projective plane, Mech. Mach. Theory, Vol.33, No.7, (1998), pp.931-944.
4. Dai S.J., An historical review of the the oretical development of rigid body displacement from Rodrigues parameters to the finite twist, Mech. Mach. Theory, Vol.41, (2006), pp.41-42.
5. Jing Li, Research on the rigid body pose estimation using dual quaternions. Advances in Mechanical Engineering, Vol.11, No.1, (2019).
6. Liao Q., Inverse kinematic analysis of the general 6R serial manipulator based on double quaternions, Mech. Mach. Theory, Vol.45, (2010), pp.193-199.

7. Mamedov F.G., Biquaternion algebra in problems of analysis and synthesis of mechanisms, Dissertation work for the degree of candidate of technical sciences, (1994).
8. Mammadov F.H., Particular drawing biquaternion closure equations of complex spatial mechanisms, Proceedings of the second International Symposium of mechanism and machine science, Baku, (2017), pp.87-92.
9. Martines J.M., Gallardo-Alvarado J., A simple method for the determination of angular velocity and acceleration of a spherical motion through quaternion, Nederland's, Meccanica, Vol.35, (2000), pp.111-118.
10. Pennestri Valentini P.P., Dual quaternions as a tool for rigid body motion analysis: a tutorial with an application to biomechanics, Multibody Dinamics, Eccomas Thematic Conference K. Arczewski, J. Frajaczek, M. Wojtyra (eds.) Warsaw, Poland, (2009), pp.1-17.
11. Thomas F., Approaching dual quaternions from matrix algebra, IEEE T Robot, V.30, 2014, pp.1037–1048.
12. Wang J.Y., Liang H.Z., Sun Z.W., et al., Relative motion coupled control based on dual quaternion, Aerosp Sci Technol, Vol.25, (2013), pp.102–113.
13. Xiao Long Yang, Hong Tao Wu, Yao Li, Bai Chen, A dual quaternion solution to the forward kinematics of a class of six-DOF parallel robots with full or reductant actuation, Mech. Mach. Theory, Vol.107, (2017), pp.27–36.
14. Zhou W., Chen W., Liu H., et al., A new forward kinematic algorithm for a general Stewart platform, Mech. Mach. Theory, Vol.87, (2015), pp.177–190.