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THE COERCIVE UNIFORM ESTIMATE FOR SOME NONLOCAL DIFFERENTIAL OPERATOR EQUATIONS

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Abstract. In this paper we study the maximal regularity properties of the Cauchy problem for the abstract nonlocal parabolic equation with parameters in weighted spaces.

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1. Introduction

In recent years, the maximal regularity properties of abstract differential equations, especially for elliptic and parabolic types have been studied extensively, e.g. in [1],[2],[4],[5],[11] and the references therein. Moreover, the nonlocal differential equations have been treated e.g. in [8]. Convolution operators in Banach–valued function spaces studied e.g. in [6],[11]. However, the nonlocal differential operator equations are relatively less investigated subjects. The parabolic type nonlocal differential equation with bounded operator coefficients was investigated in [3]. The main aim of the present paper is to establish maximal regularity properties of the Cauchy problem for the following parabolic nonlocal differential operator equations with parameter

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\le l}\varepsilon_\alpha a_\alpha * D^\alpha u + A*u = f(t,x)\,, t\in(0,T), x\in\mathbb{R}^n, \tag{1}$$

$$u(0,x)=0, x\in\mathbb{R}^n, 0<T<\infty,$$

in E – valued mixed $L_{p,\gamma}$ spaces, where $a_\alpha = a_\alpha(x)$ are complex-valued functions, l is a natural number, $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n), \alpha_k$ are nonnegative integers, $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n), \varepsilon_\alpha=\prod_{k=1}^n \varepsilon_k^{\frac{\alpha_k}{l}}, \varepsilon_k$ are positive parameter and $A=A(x)$ is a linear operator in a Banach space E. Here, the convolutions $a_\alpha * D^\alpha u, A*u$ are defined in the distribution sense (see e.g. [2]).

2. Notations and background

Let E be a Banach space and $\gamma = \gamma(x), x = (x_1, x_2, \dots, x_n)$ be a positive measurable weighted function on a measurable subset $\Omega \subset \mathbb{R}^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of all strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p \gamma(x) dx\right)^{\frac{1}{p}}, 1 \le p < \infty,$$

for $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x \in \Omega} [\gamma(x)\|f(x)\|_E].$$

The weight function $\gamma = \gamma(x)$ is said to satisfy an A_p condition, i.e., $\gamma(x) \in A_p,\ 1 < p < \infty$ if there is a positive constant C such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \gamma(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q \gamma^{-\frac{1}{p-1}}(x)\, dx\right)^{p-1} \le C$$

for all compacts $Q \subset \mathbb{R}^n$ (see $[7, Ch. 9]$.

Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda : \lambda \in \mathbb{C}, |arg\lambda| \le \varphi\} \cup \{0\}, 0 \le \varphi < \pi.$$

Let E_1 and E_2 be two Banach spaces and let $B(E_1, E_2)$ denote the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ we denote $B(E, E)$ by $B(E)$.

A closed linear operator A is said to be φ – sectorial in Banach space E with bound $M > 0$ if $KerA = \{0\}, D(A)$ and $R(A)$ are dense on E and

$$\|(A + \lambda I)^{-1}\|_{B(E)} \le M|\lambda|^{-1}$$

for all $\lambda \in S_\varphi, \varphi \in [0, \pi)$, where I is an identity operator in E. It is known that the fractional powers of the operator A are well defined. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|_E^p + \|A^\theta u\|_E^p\right)^{\frac{1}{p}}, 1 \le p < \infty, -\infty < \theta < \infty.$$

Let $S = S(\mathbb{R}^n; E)$ denotes the Schwartz class, i.e., the space of E – valued rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. Here, $S'(\mathbb{R}^n; E)$ denotes the space of all continuous linear operators $L: S(\mathbb{R}^n; E) \to E$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n;E)$ is norm dense in $L_{p,\gamma}(\mathbb{R}^n;E)$ when $1 < p < \infty,\ \gamma \in A_p$.

Let F denotes the Fourier transform defined by

$$\hat{u}(\xi) = Fu = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx \quad for\, u \in S(\mathbb{R}^n;E)$$

and $x, \xi \in \mathbb{R}^n$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} \dots (i\xi_2)^{\alpha_n} \hat{f}, D_\xi^\alpha\big(F(f)\big) = F[(-ix_1)^{\alpha_1} \dots (-ix_2)^{\alpha_n} f]$$

for all $f \in S'(\mathbb{R}^n;E)$.

The inverse Fourier transform

$$F^{-1}u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} \hat{u}(\xi)\, d\xi.$$

Let E be Banach space. The function $u \to Tu\colon \mathbb{R}^n \to B(E)$ is called a Fourier multiplier $L_{p,\gamma}(\mathbb{R}^n;E)$ for $p \in (1,\infty)$ if

$$\|F^{-1}TFu\|_{L_{p,\gamma}(\mathbb{R}^n;E)} \le C\|u\|_{L_{p,\gamma}(\mathbb{R}^n;E)}, u \in S(\mathbb{R}^n;E).$$

The space of all Fourier multipliers from $L_{p,\gamma}(\mathbb{R}^n;E)$ will be denoted $M_{p,\gamma}^{p,\gamma}(E)$.

A Banach space E is called a UMD space (see e.g. [3],[10]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is initially defined on $S(\mathbb{R};E)$ and is bounded in $L_p(\mathbb{R};E), p \in (1,\infty)$ ([5]).

A family of operators $\mathcal{T} \subset B(E_1, E_2)$ is called R – bounded if there is a constant $C > 0$ such that for all $T_1, T_2, \dots, T_k \in \mathcal{T}$ and $u_1, u_2, \dots, u_m \in E_1$ and for all independent, symmetric $\{-1;1\}$ valued random variables μ_j on $[0;1]$

$$\int_0^1 \left\| \sum_{j=1}^m \mu_j(y) T_j u_j \right\|_{E_2} dy \le C \int_0^1 \left\| \sum_{j=1}^m \mu_j(y) u_j \right\|_{E_1} dy$$

is valid. The smallest C is called the R – bound of $\mathcal{T}$ and denoted by $R(\mathcal{T})$.

Definition 2.1. A Banach space E is said to be a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$ if for any $\Psi \in C^{(n)}\big(\mathbb{R}^n \backslash \{0\}; B(E)\big)$ the R – boundedness of the set

$$\left\{|\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in \mathbb{R}^n \backslash \{0\}, \beta = (\beta_1, \beta_2, \ldots, \beta_n)\ \beta_k \in \{0,1\}\right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(\mathbb{R}^n; E)$, i.e. $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$.

Definition 2.2. A sectorial operator $A(x), x \in \mathbb{R}^n$ is said to be uniformly R – sectorial in a Banach space E if there exists $\varphi \in [0,\pi)$ such that

$$\sup_{x \in \mathbb{R}^n} R\left\{[A(x)(A(x) + \xi I)^{-1}] : \xi \in S_\varphi\right\} \leq M.$$

Let $A = A(x), x \in \mathbb{R}^n$ be closed linear operator in E with domain $D(A)$ independent x. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined

$$\hat{A}(\xi)u(\varphi) = A(x)u(\hat{\varphi})\ \ for\ u \in S'(\mathbb{R}^n; E), \varphi \in S(\mathbb{R}^n).$$

Let $A = A(x)$ be a linear operator with domain $D(A)$ independent on $x \in \mathbb{R}^n$ such that $Au \in L'(\mathbb{R}^n; E)$ for $u \in S(\mathbb{R}^n; D(A))$. The convolution $A * u$ of A and $u \in S(\mathbb{R}^n; D(A))$ is defined as

$$A * u = \int_{\mathbb{R}^n} A(x)u(x - \xi)\, d\xi \quad for\ u \in S(\mathbb{R}^n; D(A)).$$

Let E_0 and E be two Banach spaces where E_0 is continuously and densely embedded into E. Let l be a natural number. $W_{p,\gamma}^l(\mathbb{R}^n; E_0, E)$ denotes the space of all functions from $S'(\mathbb{R}^n; E_0)$ such that $u \in L_{p,\gamma}(\mathbb{R}^n; E_0)$ and the generalized derivatives $D_k^l u = \frac{\partial^l u}{\partial x_k^l} \in L_{p,\gamma}(\mathbb{R}^n; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^l(\mathbb{R}^n; E_0, E)} = \|u\|_{L_{p,\gamma}(\mathbb{R}^n; E_0)} + \sum_{k=1}^n \|D_k^l u\|_{L_{p,\gamma}(\mathbb{R}^n; E)} < \infty.$$

3. Nonlocal differential operator equations

Consider the following nonlocal differential operator equation

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u = f(x), x \in \mathbb{R}^n, \qquad (2)$$

where $A = A(x)$ is a linear operator in a Banach space E for $x \in \mathbb{R}^n$, $a_\alpha = a_\alpha(x)$ are complex – valued functions.

We defined sufficient conditions for the separability of a linear problem which are the followings.

Condition 3.1. Suppose the followings are satisfied:

(1) $L_\varepsilon(\xi) = \sum_{|\alpha|\le l} \varepsilon_\alpha \widehat{a_\alpha}(\xi)(i\xi)^\alpha \in S_{\varphi_1}, \varphi_1 \in [0,\pi)$ for $\xi \in \mathbb{R}^n$,

$$|L_\varepsilon(\xi)| \ge C \sum_{k=1}^{n} \varepsilon_k \left|\hat{a}_{\alpha(l,k)}\right| |\xi_k|^l, \alpha(l,k) = (0,0,\dots,l,0,0,\dots,0), i.e.\ \alpha_i = 0,$$
$$i \ne k, \alpha_k = l, i = 1,2,\dots,n;$$

(2) $\hat{a}_\alpha \in C^{(n)}(\mathbb{R}^n)$ and
$|\xi|^{|\beta|}\left|D^{(\beta)}\widehat{a_\alpha}(\xi)\right| \le C_1, \beta_k \in \{0,1\}, 0 \le |\beta| \le n$;

(3) for $0 \le |\beta| \le n, \xi, \xi_0 \in \mathbb{R}^n \backslash \{0\}$:
$[D^\beta \hat{A}(\xi)]\hat{A}^{-1}(\xi_0) \in C(\mathbb{R}^n; B(E)), |\xi|^{|\beta|} \left\|[D^\beta \hat{A}(\xi)]\hat{A}^{-1}(\xi_0)\right\|_{B(E)} \le C_2$.

Here $\hat{A}(\xi)$ is a uniformly φ – sectorial operator in E with $\varphi \in [0,\pi)$.

Consider operator functions

$$\sigma_{0\varepsilon}(\xi,\lambda) = \lambda D_\varepsilon(\xi,\lambda), \sigma_{1\varepsilon}(\xi,\lambda) = \hat{A}(\xi) D_\varepsilon(\xi,\lambda),$$
$$\sigma_{2\varepsilon}(\xi,\lambda) = \sum_{|\alpha|\le l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \widehat{a_\alpha}(\xi)(i\xi)^\alpha D_\varepsilon(\xi,\lambda),$$

where

$$D_\varepsilon(\xi,\lambda) = \left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1}.$$

In our old work [9] we proved the following lemma.

Lemma 3.1. Assume that Condition 3.1 is satisfied. If the operator functions $\sigma_{i\varepsilon}(\xi,\lambda)$ for $\lambda \in S_{\varphi_2}, \varphi_1 \in [0,\pi)$ and the operators $|\xi|^{|\beta|} D_\xi^\beta \sigma_{i\varepsilon}(\xi,\lambda), i = 0,1,2$ are uniformly bounded, then the following sets

$$S_{i\varepsilon}(\xi,\lambda) = \left\{|\xi|^{|\beta|} D_\xi^\beta \sigma_{i\varepsilon}(\xi,\lambda); \xi \in \mathbb{R}^n \backslash \{0\}\right\}, i = 0,1,2$$

are uniformly R – bounded for $\beta_k \in \{0,1\}, 0 \le |\beta| \le n$.

Here, E is a Banach space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$ and $\hat{A}(\xi)$ is a uniformly R – sectorial operator in E with $\varphi \in [0,\pi)$.

Now, consider the following nonlocal differential operator equation

$$\sum_{|\alpha|\le l} \varepsilon_\alpha a_\alpha * D^\alpha u + (A+\lambda) * u = f, \qquad (3)$$

where $\varepsilon, \varepsilon_\alpha, \lambda$ are parameters, a_α are complex – valued functions defined in (1) and A is a linear operator in a Banach space E.

Theorem 3.1. Assume that Condition 3.1 holds and E is a Banach space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$. Let $\hat{A}(\xi)$ be a uniformly R – sectorial operator in E with $\varphi \in [0,\pi)$.

$\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$. Then, problem (3) has a unique solution u and the coercive uniform estimate holds

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X + \|A * u\|_X + |\lambda| \|u\|_X \leq C\|f\|_X \quad (4)$$

for all $f \in X$ and $\lambda \in S_\varphi$.

Here,

$$X = L_{p,\gamma}(\mathbb{R}^n; E), Y = W^l_{p,\gamma}(\mathbb{R}^n; E(A), E), p \in (1, \infty).$$

Let Φ_ε be an operator in X generated by problem (3) for $\lambda = 0$, i.e.

$$D(\Phi_\varepsilon) \subset Y, \Phi_\varepsilon u = \sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u.$$

Theorem 3.2. Assume that Theorem 3.1 holds and the following conditions are satisfied:

1) $C_1 \|\hat{A}(\xi_0)u\|_E \leq \|A(x)u\|_E \leq C_2 \|\hat{A}(\xi_0)u\|_E, \xi_0 \in \mathbb{R}^n, u \in D(A), x \in \mathbb{R}^n;$
2) for $\alpha(l,k) = (0,0,\ldots,l,0,0,\ldots,0)$, i.e. $\alpha_i = 0,\ i \neq k, \alpha_k = l$,

$$C_1 \sum_{k=1}^{n} \varepsilon_k |\hat{a}_{\alpha(l,k)}| |\xi_k|^l \leq |L_\varepsilon(\xi)| \leq C_2 \sum_{k=1}^{n} \varepsilon_k |\hat{a}_{\alpha(l,k)}| |\xi_k|^l, \xi \in \mathbb{R}^n,$$

and there exists $x_0 \in \mathbb{R}^n$ such that

$$\hat{A}(\xi) A^{-1}(x_0) \in L_\infty(\mathbb{R}^n; B(E)), \xi, x_0 \in \mathbb{R}^n,$$

$$C_1 \|A(x_0)u\| \leq \|A(x)u\| \leq C_2 \|A(x_0)u\|, u \in D(A), x \in \mathbb{R}^n$$

where C_1, C_2 are positive constants.

Then for $u \in Y$ there are positive constants M_1, M_2 such that

$M_1 \|u\|_Y \leq \|\Phi_\varepsilon u\|_X \leq M_2 \|u\|_Y$.

From Theorem 3.1 we have:

Result 3.1. Assume that the all conditions of Theorem 3.1 are satisfied. Then, for all $\lambda \in S_{\varphi_2}$ the following uniform coercive estimate holds

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha (\Phi_\varepsilon + \lambda)^{-1}\|_{B(X)} + \|A * (\Phi_\varepsilon + \lambda)^{-1}\|_{B(X)}$$
$$+ \|\lambda (\Phi_\varepsilon + \lambda)^{-1}\|_{B(X)} \leq C.$$

4. The Cauchy problem for parabolic nonlocal differential operator equations

In this section, we shall consider the Cauchy problem for the nonlocal parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\le l}\varepsilon_\alpha a_\alpha * D^\alpha u + A*u = f(t,x), \qquad (5)$$

$$u(0,x)=0, \qquad t\in(0,T), x\in\mathbb{R}^n, T<\infty$$

where ε_α is defines as in (1) a_α are complex valued functions defined as in (1) and A is a linear operator in a Banach space E.

By using the definition of the norm of the function for the space $L_{\mathbf{p}}(R^n), \mathbf{p}=(p_1,p_2,\dots,p_n)$ which the norm is

$$\|f\|_{\mathbf{p},R^n} = \|f\|_{(p_1,p_2,\dots,p_n),R^n}$$

$$=\left\{\int_{R^n}\left[\dots\left\{\int_{R^2}\left(\int_{R^1}|f(x)|^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\right\}^{\frac{p_3}{p_2}}\dots\right]^{\frac{p_n}{p_{n-1}}}dx_n\right\}^{\frac{1}{p_n}},$$

we can be denoted the space of all $\mathbf{p}$-summable E – valued functions for $\mathbb{R}_T^{n+1}=(0,T)\times\mathbb{R}^n, \mathbf{p}=(p,p_1)$, $Z=L_{\mathbf{p},\gamma}(\mathbb{R}_T^{n+1};E)$ with mixed norm, which is

$$\|f\|_Z=\left(\int_{\mathbb{R}^n}\left(\int_0^T\|f(x,t)\|_E^p\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}<\infty.$$

Here, the space of all measurable E – valued functions f defined on $\mathbb{R}_T^{n+1}$.

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Suppose l is an integer and $Z_0=W_{\mathbf{p},\gamma}^{1,l}(\mathbb{R}_T^{n+1};E_0,E)$ denotes the space of all functions $u\in Z$ such that the generalized derivatives $D_t u, D_k^l u\in Z$, with the norm

$$\|u\|_{Z_0}=\|u\|_{Z(E_0)}+\|D_t u\|_Z+\sum_{k=1}^n\left\|D_k^l u\right\|_Z,$$

where

$$Z(E_0)=L_{\mathbf{p},\gamma}(\mathbb{R}_T^{n+1};\ E_0)$$

Applying Theorem 3.1 we establish the maximal regularity of (5) in *Z*. For this purpose, we need the following result:

Theorem 4.1. Assume that the all conditions of Theorem 3.1 are satisfied. Then operator Φ_ε is uniformly R −sectorial in X.

Proof. The Result 3.1 implies that Φ_ε is a sectorial operator in X. We have to prove the R −boundedness of the set

$\sigma_\varepsilon(\xi,\lambda) = \{\lambda(\Phi_\varepsilon+\lambda)^{-1} : \lambda \in S_\varphi\}$.

Indeed, from the proof of Theorem 3.1 we have

$\lambda(\Phi_\varepsilon+\lambda)^{-1} f = F^{-1}\sigma_{0\varepsilon}(\xi,\lambda)\hat{f}, f \in X,$

where

$\sigma_{0\varepsilon}(\xi,\lambda) = \lambda\left[\hat{A}(\xi)+L_\varepsilon(\xi)+\lambda\right]^{-1}$.

By using Lemma 3.1 and definition of R −boundedness, it is enough to show that the operator function $\sigma_{0\varepsilon}(\xi,\lambda)$ (depended on variable λ and parameter ξ) is a multiplier in X. Then, we have

$$\int_0^1 \left\| \sum_{j=1}^m \mu_j(y)\lambda_j(\Phi_\varepsilon+\lambda_j)^{-1} f_j \right\|_X dy$$

$$= \int_0^1 \left\| \sum_{j=1}^m \mu_j(y) F^{-1}\sigma_{0\varepsilon}(\xi,\lambda_j)\hat{f}_j \right\|_X dy$$

$$= \int_0^1 \left\| F^{-1} \sum_{j=1}^m \mu_j(y)\sigma_{0\varepsilon}(\xi,\lambda_j)\hat{f}_j \right\|_X dy$$

$$\le C \int_0^1 \left\| \sum_{j=1}^m \mu_j(y) f_j \right\|_X dy$$

for all $\xi \in \mathbb{R}^n, \lambda_1,\lambda_2,\dots,\lambda_m \in S_\varphi, f_1,f_2,\dots,f_m \in X, m \in \mathbb{N}$ where $\{\mu_j\}$ is a sequence of independent symmetric $\{-1;1\}$ valued random variables on $[0;1]$. Hence, the set $\sigma_\varepsilon(\xi,\lambda)$ is uniformly R −bounded.

Now, we are ready to state the main result of this section.

Theorem 4.2. Assume that all the conditions of Theorem 3.1 are satisfied for $\varphi \in \left(\frac{\pi}{2},\pi\right)$. Then the equation (5) has a unique solution $u \in W_{p,\gamma}^{1,l}(\mathbb{R}_T^{n+1}; E(A), E)$. Moreover, the following coercive uniform estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_Z + \sum_{|\alpha|\le l} \varepsilon_\alpha \|a_\alpha * D^\alpha u\|_Z + \|A*u\|_Z \le C\|f\|_Z. \tag{6}$$

Proof. By Fubini's theorem we have $Z = L_{p1}(0,T;X)$. Moreover, by definition of spaces Y, Z_0 for $E_0 = E(A)$ and by Theorem 3.2 we obtain

$$\|u\|_{Z_0} = \|u\|_{Z(A)} + \left\|\frac{du}{dt}\right\|_{L_{p1}(0,T;X)} + \|\Phi_\varepsilon u\|_{L_{p1}(0,T;X)} \simeq \left\|\frac{\partial u}{\partial t}\right\|_Z + \|\Phi_\varepsilon u\|_Z$$

$$\simeq \|Au\|_Z + \left\|\frac{\partial u}{\partial t}\right\|_Z + \sum_{k=1}^{n} \left\|D_k^l u\right\|_Z \simeq \|u\|_{Z_0}$$

where

$Z(A) = L_{p1}\left(0,T;X(A)\right), X(A) = L_{p,\gamma}\left(\mathbb{R}_T^n;E(A)\right), X = L_{p,\gamma}(\mathbb{R}^n;E).$

Hence, we get

$Z_0 = W_{p1}^1(0,T;D(\Phi_\varepsilon),X), for\ E_0 = E(A).$

Therefore, the problem (5) can be expressed as

$$\frac{du}{dt} + \Phi_\varepsilon u(t) = f(t), u(0) = 0, t \in \mathbb{R}_+. \tag{7}$$

By virtue of $[2, Theorem\ 4.5.2]$ and $[6]$, X is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$. Then due to R −sectoriality of Φ_ε with $\varphi \in \left(\frac{\pi}{2},\pi\right)$, by virtue of $[12, Theorem\ 4.2]$, for $f \in L_{p1}(0,T;X)$ the problem

$$\left\|\frac{du}{dt}\right\|_{L_{p1}(0,T;X)} + \|\Phi_\varepsilon u\|_{L_{p1}(0,T;X)} \le C\|f\|_{L_{p1}(0,T;X)}.$$

In view of Results 3.1 and from the above estimate, we get (6).

5. Conculusion

As a result, we obtained the coercive uniform estimate of the solution of the Cauchy problem for parabolic nonlocal differential operator equations.

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